1. Numerical Differentiation (25): The Taylor series of a function $f(x)$ around $x_i$ is given by $T(x) = f(x_i) + f'(x_i) * (x - x_i) + \frac{1}{2!} f''(x_i) * (x - x_i)^2 + ....$

   (a) Derive the statement of the forward difference approximation of the derivative $f'(x_i)$.
   *hint:* your expression should converge to $f'(x_i)$ as $(x_{i+1} - x_i) \to 0$

   (b) Given the heat equation $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$, with the discrete state $\{u_i^n\}$ denoted by $u^n$, express the matrix used for the forward Euler approximation of the solution (that is, the $A$ in $u^{n+1} = Au^n$ from the homework).

2. Numerical Integration (25)

   (a) Given a continuous function $f(x)$ with domain $[a, b]$ with uniform partition $\{x_0, x_1, ..., x_n\}$, derive the Trapezoid rule to estimate the value of the integral $\int_a^b f(x)\ dx$

   (b) Estimate the integral $\int_{-1}^{1} x^4 + x^2 + 1\ dx$ using the partition $\{-1, 0, 1\}$ and the Trapezoid rule.

   (c) Estimate that same integral using 3-point Gauss-Legendre Quadrature. The points and weights are as follows:

| $x_i$ | $w_i$ |
|---|---|
| $-\sqrt{\frac{3}{5}}$ | $\frac{5}{9}$ |
| $0$ | $\frac{8}{9}$ |
| $\sqrt{\frac{3}{5}}$ | $\frac{5}{9}$ |

3. Numerical Solutions to ODEs (25)

   (a) Consider the differential equation $f(x, u(x)) = x + u(x)$ with $u(0) = 1$. Using a step size of $\Delta x = 1$, compute an approximation to $u(5)$ using any technique we covered in class.x

4. Topic (10): Let $f(x)$ be a continuous function on the domain $[a, b]$; let $\{x_{2n}\}$ be a uniform partition of $[a, b]$ into $2n$ intervals such that $x_0 = a$ and $x_{2n} = b$ and $x_{i+1} - x_i = \Delta x$.

   (a) Write down Simpson's rule for numerical integration of $f$ using $2n$ intervals.

   (b) Write down the Trapezoid rule for numerical integration of $f$ using the partition $x_0, x_2, ..., x_{2n}$ (the $n$ even-indexed points form the partition here).

   (c) Write down the Midpoint method for numerical integration of $f$, again starting from the partition $x_0, x_2, ..., x_{2n}$.

   (d) If $S_{2n}$ is the Simpson's estimate for the integral of $f$, $T_n$ is the Trapezoid estimate over only the even points, and $M_n$ is the Midpoint estimate over only the odd points, prove that $S_{2n} = \frac{1}{3}(T_n + 2M_n)$.

1. Numerical Differentiation (25): The Taylor series of a function $f(x)$ around $x_i$ is given by $T(x) = f(x_i) + f'(x_i) * (x - x_i) + \frac{1}{2!}f''(x_i) * (x - x_i)^2 + \frac{1}{3!}f'''(x_i) * (x - x_i)^3 + ....$
   (a) Derive the statement of the central difference approximation of the second derivative $f''(x_i)$.
   *hint:* your expression should converge to $f''(x_i)$ as $(x_{i+1} - x_i) \to 0$
   (b) Given the heat equation $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$, with the discrete state $\{u_i^n\}$ denoted by $u^n$, express the matrix used for the backwards Euler approximation of the solution (that is, the $A$ in $u^{n-1} = Au^n$ from the homework).

2. Numerical Integration (25)
   (a) Given a continuous function $f(x)$ with domain $[a, b]$ with uniform partition $\{x_0, x_1, ..., x_n\}$, derive the Midpoint rule to estimate the value of the integral $\int_a^b f(x)\ dx$
   (b) Estimate the integral $\int_{-1}^{1} x^4 + x^2 + 1\ dx$ using the partition $\{-1, 0, 1\}$ and the Midpoint rule.
   (c) Estimate that same integral using 3-point Gauss-Legendre Quadrature. The points and weights are as follows:

| $x_i$ | $w_i$ |
|---|---|
| $-\sqrt{\frac{3}{5}}$ | $\frac{5}{9}$ |
| $0$ | $\frac{8}{9}$ |
| $\sqrt{\frac{3}{5}}$ | $\frac{5}{9}$ |

3. Numerical Solutions to ODEs (25)
   (a) Consider the differential equation $f(x, u(x)) = (x - u(x))x$ with $u(0) = 1$. Using a step size of $\Delta x = 1$, compute an approximation to $u(5)$ using any technique we covered in class.

4. Topic (10): Let $f(x)$ be a twice-differentiable function on the domain $[a, b]$ and furthermore suppose that for all $x \in [a, b]$ that $f''(x) > 0$.
   (a) Write down (you do not need to derive) $\hat{f}$, the forward difference estimation for $f(b)$.
   (b) Write down (you do not need to derive) $\bar{f}$, the backwards difference estimation for $f(b)$.
   (c) Prove $\hat{f} < \bar{f}$.
   (d) For the true value of the function $f(b)$, prove $f(b)$ is bounded by $\hat{f}$ and $\bar{f}$.
   *hint: employ the Mean Value Theorem in a proof by contradiction:* If $f(x)$ is differentiable on $[a, b]$ and continuous on $(a, b)$, then $\exists c \in (a, b)\colon f'(c) = \frac{f(b) - f(a)}{b - a}$.