- 1. Numerical Differentiation (25): The Taylor series of a function f(x) around  $x_i$  is given by  $T(x) = f(x_i) + f'(x_i) * (x - x_i) + \frac{1}{2!}f''(x_i) * (x - x_i)^2 + \dots$ 
  - (a) Derive the statement of the forward difference approximation of the derivative  $f'(x_i)$ . *hint:* your expression should converge to  $f'(x_i)$  as  $(x_{i+1} - x_i) \to 0$
  - (b) Given the heat equation  $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$ , with the discrete state  $\{u_i^n\}$  denoted by  $u^n$ , express the matrix used for the forward Euler approximation of the solution (that is, the A in  $u^{n+1} = Au^n$  from the homework).
- 2. Numerical Integration (25)
  - (a) Given a continuous function f(x) with domain [a, b] with uniform partition  $\{x_0, x_1, ..., x_n\}$ , derive the Trapezoid rule to estimate the value of the integral  $\int_a^b f(x) dx$
  - (b) Estimate the integral  $\int_{-1}^{1} x^4 + x^2 + 1 \, dx$  using the partition  $\{-1, 0, 1\}$  and the Trapezoid rule.
  - (c) Estimate that same integral using 3-point Gauss-Legendre Quadrature. The points and weights are as follows:

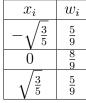
$x_i$	$w_i$
$-\sqrt{\frac{3}{5}}$	$\frac{5}{9}$
0	$\frac{8}{9}$
$\sqrt{\frac{3}{5}}$	$\frac{5}{9}$

- 3. Numerical Solutions to ODEs (25)
  - (a) Consider the differential equation f(x, u(x)) = x + u(x) with u(0) = 1. Using a step size of  $\Delta x = 1$ , compute an approximation to u(5) using any technique we covered in class.x
- 4. Topic (10): Let f(x) be a continuous function on the domain [a, b]; let  $\{x_{2n}\}$  be a uniform partition of [a, b] into 2n intervals such that  $x_0 = a$  and  $x_{2n} = b$  and  $x_{i+1} x_i = \Delta x$ .
  - (a) Write down Simpson's rule for numerical integration of f using 2n intervals.
  - (b) Write down the Trapezoid rule for numerical integration of f using the partition  $x_0, x_2, ..., x_{2n}$  (the *n* even-indexed points form the partition here).
  - (c) Write down the Midpoint method for numerical integration of f, again starting from the partition  $x_0, x_2, ..., x_{2n}$ .
  - (d) If  $S_{2n}$  is the Simpson's estimate for the integral of f,  $T_n$  is the Trapezoid estimate over only the even points, and  $M_n$  is the Midpoint estimate over only the odd points, prove that  $S_{2n} = \frac{1}{3} (T_n + 2M_n)$ .

- 1. Numerical Differentiation (25): The Taylor series of a function f(x) around  $x_i$  is given by  $T(x) = f(x_i) + f'(x_i) * (x - x_i) + \frac{1}{2!}f''(x_i) * (x - x_i)^2 + \frac{1}{3!}f'''(x_i) * (x - x_i)^3 + \dots$ 
  - (a) Derive the statement of the central difference approximation of the second derivative  $f''(x_i)$ . *hint:* your expression should converge to  $f''(x_i)$  as  $(x_{i+1} - x_i) \to 0$

(b) Given the heat equation  $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$ , with the discrete state  $\{u_i^n\}$  denoted by  $u^n$ , express the matrix used for the backwards Euler approximation of the solution (that is, the A in  $u^{n-1} = Au^n$  from the homework).

- 2. Numerical Integration (25)
  - (a) Given a continuous function f(x) with domain [a, b] with uniform partition  $\{x_0, x_1, ..., x_n\}$ , derive the Midpoint rule to estimate the value of the integral  $\int_a^b f(x) dx$
  - (b) Estimate the integral  $\int_{-1}^{1} x^4 + x^2 + 1 \, dx$  using the partition  $\{-1, 0, 1\}$  and the Midpoint rule.
  - (c) Estimate that same integral using 3-point Gauss-Legendre Quadrature. The points and weights are as follows:



- 3. Numerical Solutions to ODEs (25)
  - (a) Consider the differential equation f(x, u(x)) = (x u(x))x with u(0) = 1. Using a step size of  $\Delta x = 1$ , compute an approximation to u(5) using any technique we covered in class.
- 4. Topic (10): Let f(x) be a twice-differentiable function on the domain [a, b] and furthermore suppose that for all  $x \in [a, b]$  that f''(x) > 0.
  - (a) Write down (you do not need to derive)  $\hat{f}$ , the forward difference estimation for f(b).
  - (b) Write down (you do not need to derive)  $\overline{f}$ , the backwards difference estimation for f(b).
  - (c) Prove  $\hat{f} < \bar{f}$ .
  - (d) For the true value of the function f(b), prove f(b) is bounded by f̂ and f̄.
    hint: employ the Mean Value Theorem in a proof by contradiction: If f(x) is differentiable on [a, b] and continuous on (a, b), then ∃c ∈ (a, b): f'(c) = f(b) f(a)/b a.