Do 4 of these 6 problems. Only the four highest scoring answers will be taken for the test score. Existing results must be quoted as coming from either homework, class, or the textbook. *The validity of your reasoning matters far more than your ability to attain the correct answer!*

1. Suppose $b \in \mathbb{R}$. Show that the set of continuous real-valued functions f on the interval $[0, 1]$ such that $\int_0^1 f = b$ is a subspace of $\mathbb{R}^{[0,1]}$ if and only if $b = 0$.

2. Prove that $\mathbb{R}^\infty$ is infinite-dimensional.

3. Suppose $T \in \mathcal{L}(V, W)$, and $w_1, ..., w_m$ is a basis of range T. Prove that there exist $\varphi_1, ..., \varphi_m \in \mathcal{L}(V, \mathbb{F})$ such that

$$Tv = \varphi_1(v)w_1 + \cdots + \varphi_m(v)w_m$$

   for every $v \in V$.

4. Suppose V is finite-dimensional and $v_1, ..., v_m \in V$. Define a linear map $\Gamma \colon \mathcal{L}(V, \mathbb{F}) \to \mathbb{F}^m$ by

$$\Gamma(\varphi) = \big(\varphi(v_1), ..., \varphi(v_m)\big).$$

   (a) Prove that $v_1, ..., v_m$ spans V if and only if Γ is injective.

   (b) Prove that $v_1, ..., v_m$ is linearly independent if and only if Γ is surjective.

5. Find all eigenvalues and eigenvectors of the backwards shift operator $T \in \mathcal{L}(\mathbb{F}^\infty)$ defined by

$$T(z_1, z_2, z_3, ...) = (z_2, z_3, ...)$$

6. Suppose $T \in \mathcal{V}$ and $U \subset V$ is a subspace of V invariant under T. Prove or give a counterexample: every subspace of U is invariant under T.