Problem 6. Integrability of f on $\mathbb{R}$ does not necessarily imply the convergence of $f(x)$ to 0 as $x \to \infty$.

(a) There exists a positive continuous function f on $\mathbb{R}$ so that f is integrable on $\mathbb{R}$, but yet $\limsup_{x\to\infty} f(x) = \infty$.

(b) However, if we assume that f is uniformly continuous on $\mathbb{R}$ and integrable, then $\lim\limits_{|x|\to\infty} f(x) = 0$.

(a) The function that satisfies this condition is, in short, triangles of increasing height but with decreasing area. Specifically, the function we will use is

$$f(x) = \left\{ \begin{array}{cl} 0 & x \leq n \\ n^4x - n^5 & n < x \leq n + \frac{1}{n^3} \\ -n^4x + 2n + n^5 & n + \frac{1}{n^3} < x \leq n + \frac{2}{n^3} \\ 0 & n + \frac{2}{n^3} < x \end{array} \right\} \text{ for all } n \geq 2, n \in \mathbb{N}$$

This function is zero up to n, then it increases linearly to n to the point $(n + \dfrac{1}{n^3}, n)$, then decreases linearly to the point $(n + \dfrac{2}{n^3}, 0)$; the end result is the graph looks like a sequence of peaks, each of which whose integral is $\frac{1}{n^2}$ and whose height is n. f is positive, and f is continuous as each of the pieces have been carefully chosen to agree at their endpoints (and linear functions are continuous). The integral of f is the area under the graph: in other words, $\displaystyle\int_{\mathbb{R}} f = \sum_{n=2}^{\infty} \frac{1}{n^2}$. This is a known convergent series, whose sum is $\dfrac{\pi^2}{6} - 1$; hence f is integrable. However, since f is unbounded (since $f(n + \dfrac{1}{n^3}) = n$ for all $n \geq 2, n \in \mathbb{N}$) we conclude that $\limsup_{x\to\infty} f(x) = \infty$.

(b) *Claim:* f uniformly continuous on $\mathbb{R}$, f integrable on $\mathbb{R}$ $\implies \lim\limits_{|x|\to\infty} f(x) = 0$

Proof. Assume for sake of contradiction that $\lim\limits_{|x|\to\infty} f(x) \neq 0$. We already know that for the general case f is said to be integrable if $|f|$ is integrable in the nonnegative sense of Lebesgue integrability. Hence if we can show $|f| = g$ is not integrable under this assumption, that f is not integrable under this assumption- and thus, we would have our contradiction. If f is uniformly continuous then g is also uniformly continuous, since $|g(x) - g(y)| = \big||f(x)| - |f(y)|\big| \leq |f(x) - f(y)|$ by the triangle inequality. Without loss of generality (since we can always use $f(-x)$; $f(x)$ uniformly continuous and integrable means $f(-x)$ is also uniformly continuous and integrable) we can rewrite this assumption as $\lim\limits_{x\to\infty} f(x) \neq 0$.

Thus, by assuming the limit is not zero, there exists an $\epsilon > 0$ such that for all $k > 0$, there exists an $x > k$ such that $|f(x)| = g(x) > \epsilon$. We will use the instances where k is a positive integer, and we will label the x_k we get with this hypothesis so that we now have a countable sequence $\{x_k\}$ such that $g(x_k) > \epsilon$. Note that it is entirely possible for x_k and x_{k+1} to be the same point; however, it is impossible for this sequence to be bounded. If $\{x_k\}$ were bounded, then there would be a largest member x^*: if we let $m > x^*$ be the next largest positive integer, the assumption above guarantees the existence of an x_m such that $x_m > m$ and $g(x_m) > \epsilon$; but $x_m > m > x^*$.

By assuming the limit is not zero, we are given an $\epsilon > 0$ and an infinite collection $\{x_k\}$ such that $g(x_k) > \epsilon$. As a matter of housekeeping, call $\{\tilde{x}_k\}$ the collection of unique x_k; thus, while $\{x_k\}$ may have duplicate elements, $\{\tilde{x}_k\}$ does not. Using this same ϵ, we make use of the hypothesis of g's uniform continuity on $\frac{\epsilon}{2}$: there exists a $\delta > 0$ such that $|\tilde{x}_k - x| < \delta \implies |g(\tilde{x}_k) - g(x)| < \frac{\epsilon}{2}$ for every $\tilde{x}_k$. Two cases:

(1) $g(\tilde{x}_k) \geq g(x)$
In which case

$$\frac{\epsilon}{2} > |g(\tilde{x}_k) - g(x)| = g(\tilde{x}_k) - g(x)$$
$$g(x) > g(\tilde{x}_k) - \frac{\epsilon}{2}; \qquad g(\tilde{x}_k) > \epsilon \text{ so}$$
$$g(x) > \frac{\epsilon}{2} \text{ for } x : |\tilde{x}_k - x| < \delta$$

(2) $g(\tilde{x}_k) < g(x)$
In which case $g(x) > g(\tilde{x}_k) > \epsilon > \frac{\epsilon}{2}$ for $x : |\tilde{x}_k - x| < \delta$.

Thus, in general for every $\tilde{x}_k$ the value of $g(x)$ on the ball $B_\delta(\tilde{x}_k)$ of radius δ centered at $\tilde{x}_k$ is bounded below by $\frac{\epsilon}{2}$. Hence the integral of g on every one of these $B_\delta(\tilde{x}_k)$ is bounded below by the integral of $\frac{\epsilon}{2}$ on the same $B_\delta(\tilde{x}_k)$ (monotonicity of the integral; $g \geq 0$ and $\frac{\epsilon}{2} > 0$). Thus, since the integral of the constant function is easily evaluated,

$$\int_{B_\delta(\tilde{x}_k)} g \geq 2\delta \cdot \frac{\epsilon}{2} = \delta\epsilon$$

The key detail is that ϵ was given at the start by assuming for sake of contradiction, and that δ was furnished with respect to ϵ only. Thus, this result is true at every one of the $\tilde{x}_k$ points. Call the set $B = \bigcup_{i=1}^{\infty} B_i$ the countable union of all the $B_\delta(\tilde{x}_k)$ balls which are all disjoint with each other. Since there are countably many unique $\tilde{x}_k$ and $\tilde{x}_k$ is unbounded, for any δ we will have still have countably many, now disjoint $B_\delta(\tilde{x}_k)$.

$\int g = \int_B g + \int_{B^c} g$ by additivity. g is nonnegative, so this can be rewritten as $\int g \geq \int_B g$ via monotonicity ($g \geq 0 \implies \int_{B^c} g \geq 0$). However, the integral over B of g is the integral of g over countably many disjoint $B_\delta(\tilde{x}_k)$- since they are disjoint, we apply additivity to get that

$$\begin{aligned}\int g \geq \int_B g &= \sum_{i=1}^{\infty} \int_{B_i} g \\ &\geq \sum_{i=1}^{\infty} \delta\epsilon\end{aligned}$$

Observe that $\delta\epsilon > 0$ is just some positive constant, so that sum diverges- hence $\displaystyle\int_B g = \infty$, and hence g is not integrable, and hence f is not integrable, contradicting our original assumptions. Thus $\lim_{|x|\to\infty} f(x) = 0$ $\square$

Problem 8. If f is integrable on $\mathbb{R}$, show that $F(x) = \int_{-\infty}^{x} f(t)\, dt$ is uniformly continuous.

Proof. Let f be integrable on $\mathbb{R}$, let $\epsilon > 0$. By Proposition 1.12(ii), there must exist a $\delta > 0$ such that for E a measurable set,

$$\int_E |f| < \epsilon \qquad \text{when } m(E) < \delta$$

Thus, for any interval in $\mathbb{R}$ with endpoints x, y, $x < y$ we have $|x - y| < \delta$; thus, we can apply 1.12(ii) on this interval and conclude

$$\int_x^y |f| < \epsilon \qquad \text{for } |x - y| < \delta$$

Evaluate $|F(x) - F(y)| = |F(y) - F(x)|$:

$$\begin{aligned} |F(y) - F(x)| &= \left| \int_{-\infty}^{y} f - \int_{-\infty}^{x} f \right| \text{ by definition} \\ &= \left| \int_x^y f \right| \text{ a consequence of additivity*} \\ &\leq \int_x^y |f| \text{ triangle inequality} \\ &< \epsilon \end{aligned}$$

Hence for any $\epsilon > 0$ there exists $\delta > 0$ such that for any x, y with $|x - y| < \delta$ we have $|F(x) - F(y)| < \epsilon$; F is uniformly continuous.

* $\int_{E \cup F} f - \int_E f = \int_f f$ for E, F disjoint. □

Problem 9. Chebyshev inequality. Suppose $f \geq 0$, and f is integrable. If $\alpha > 0$ and $E_\alpha = \{x : f(x) > \alpha\}$, prove that

$$m\left(E_\alpha\right) \leq \frac{1}{\alpha} \int f.$$

Proof. Let $\alpha > 0$ and $E_\alpha = \{x : f(x) > \alpha\}$. Decompose $\displaystyle\int f$ into $\displaystyle\int_{E_\alpha} f + \int_{E_\alpha^c} f$. f is nonnegative, and the integral of a nonnegative function is nonnegative. Since f is integrable, both of these decomposed integrals must therefore be finite and nonnegative (else $\displaystyle\int f$ would be infinite). Now:

$$\begin{aligned}
\int f &= \int_{E_\alpha} f + \int_{E_\alpha^c} f \\
&\geq \alpha \cdot m(E_\alpha) + \int_{E_\alpha^c} f \text{ by definition of } x \in E_\alpha\text{: } f(x) > \alpha\text{; monotonicity} \\
\frac{1}{\alpha}\int f &\geq m(E_\alpha) + \frac{1}{\alpha}\int_{E_\alpha^c} f \geq m(E_\alpha) \text{ since } f \text{ is nonnegative and } \alpha > 0 \\
\frac{1}{\alpha}\int f &\geq m(E_\alpha)
\end{aligned}$$

□

Problem 11. Prove that if f is integrable on $\mathbb{R}^d$, real-valued, and $\displaystyle\int_E f(x)\ dx \geq 0$ for every measurable E, then $f(x) \geq 0$ a.e. x. As a result, if $\displaystyle\int_E f(x)\ dx = 0$ for every measurable E, then $f(x) = 0$ a.e.

Proof. Let A be the set $\{x : f(x) < 0\}$. The condition $f \geq 0$ a.e. amounts to a claim that the points where $f < 0$ are a set of measure zero, that is, $m(A) = 0$. Assume for sake of contradiction that $m(A) \neq 0$; $m(A)$ is some nonzero value. Thus for $x \in A$ we have $f(x) < 0$; so $-f(x) > 0$ and therefore $\displaystyle\int_A -f(x) > 0$ by monoticity (since $m(A) > 0$ and $-f(x) > 0$ for $x \in A$ this integral cannot be zero). By linearity this means $-\displaystyle\int_A f(x) > 0$; $\displaystyle\int_A f(x) < 0$. However, we have as an assumption that $\displaystyle\int_E f(x) \geq 0$ for any measurable set E. This is a contradiction, thus $f \geq 0$ a.e.

Now, if we claim $\displaystyle\int_E f(x)\ dx = 0$ for every measurable E: first, we can apply the result just proven. $\displaystyle\int_E f(x)\ dx = 0 \geq 0$ for every measurable E, so $f(x) \geq 0$ a.e. x. Second, we can do the same thing, except to $-f(x)$: since $\displaystyle\int_E f(x)\ dx = 0$, $\displaystyle\int_E -f(x)\ dx = 0$ as well. Hence $\displaystyle\int_E -f(x)\ dx = 0 \geq 0$ for every measurable E, so $-f(x) \geq 0$ a.e. x as well. Thus $f(x) \geq 0 \geq f(x)$ a.e., and so $f(x) = 0$ a.e. $\square$

Problem 15. Consider the function defined over $\mathbb{R}$ by

$$f(x) = \begin{cases} x^{-1/2} & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

For a fixed enumeration $\{r_n\}_{n=1}^{\infty}$ of the rationals $\mathbb{Q}$, let

$$F(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n).$$

Prove that F is integrable, hence the series defining F converges for almost every $x \in \mathbb{R}$. However, observe that this series is unbounded on every interval, and in fact, any function $\tilde{F}$ that agrees with F a.e. is unbounded in any interval.

We will evaluate $\int_{\mathbb{R}} F(x)\ dx = \int_{\mathbb{R}} \sum_{n=1}^{\infty} 2^{-n} f(x - r_n)\ dx$. Note that the function $a_n(x) = 2^{-n} f(x - r_n)$ is nonnegative (f is nonnegative, being zero or the square root reciprocal) and for every $n \geq 1$, $f(x - r_n)$ is measurable*). Using Corollary 1.10, we can exchange the integral and summation processes, yielding

$$\begin{aligned}
\int_{\mathbb{R}} F(x)\ dx &= \int_{\mathbb{R}} \sum_{n=1}^{\infty} 2^{-n} f(x - r_n)\ dx = \sum_{n=1}^{\infty} \int_{\mathbb{R}} 2^{-n} f(x - r_n)\ dx \\
&= \sum_{n=1}^{\infty} 2^{-n} \int_{\mathbb{R}} f(x - r_n)\ dx \\
&= \sum_{n=1}^{\infty} 2^{-n} \int_{r_n}^{1 - r_n} f(x - r_n)\ dx \text{ because } f = 0 \text{ outside } (0,1) \\
&= \sum_{n=1}^{\infty} 2^{-n} \int_{0}^{1} f(u)\ du \text{ with the } u\text{-substitution } u = x - r_n \\
&= \sum_{n=1}^{\infty} 2^{-n} \int_{0}^{1} u^{-1/2}\ du \text{ since now we only have the interval 0 to 1} \\
&= \sum_{n=1}^{\infty} 2^{-n} \cdot 2 \cdot u^{1/2}\Big|_0^1 \\
&= \sum_{n=1}^{\infty} 2^{-n} \cdot 2 \\
&= 2
\end{aligned}$$

Hence $\int_{\mathbb{R}} F(x)\ dx = 2$; again, with an application of the results of Corollary 1.10, if $\int_{\mathbb{R}} F(x)\ dx$ is finite we immediately have that the series $\sum_{n=1}^{\infty} a_n(x)$ converges for almost every x.

Let (v, w) be any interval in $\mathbb{R}$, and let $t \in (v, w)$. Consider the value of $F(t)$: $F(t) = \sum_{n=1}^{\infty} 2^{-n} f(t - r_n)$. For any $t \in \mathbb{R}$ we can find a sequence of rational numbers approaching t from the right. Thus the term on the inside of the series, $2^{-n} f(t - r_n) = \dfrac{2^{-n}}{(t - r_n)^{1/2}}$ does not have a bound and thus the series as a whole is unbounded for any interval. If $\tilde{F}$ is a function that agrees with F a.e. then on any interval with nonzero measure there must be a point t^* such that $F(t^*) = \tilde{F}(t^*)$ (else F and $\tilde{F}$ would not agree on a set of measure greater than zero); but since we showed earlier that the series $F(t^*)$ stands for is unbounded everywhere, $\tilde{F}(t^*)$ must also be unbounded on any interval with nonzero measure.

(*): $f(x - r_n)$ is measurable: For all $\alpha > 0$ the set $E = \{x : f(x - r_n) > \alpha\}$ is the set of values x where $(x - r_n)^{\frac{1}{2}} > \alpha$. By definition for x outside $(r_n, 1 + r_n)$ the function has zero value and hence is not in E. In other words, E is the set of values where $\dfrac{1}{\alpha^2} > x - r_n$; $r_n < x < 1 + r_n$. Thus the measure of E is the measure of the open interval between the points 0 and $\dfrac{1}{\alpha^2} + r_n$; in other words, E is measurable for any $\alpha > 0$, and thus $f(x - r_n)$ is measurable.

Problem 4. Prove that if f is integrable on $\mathbb{R}^d$, and f is not identically zero, then

$$f^*(x) \geq \frac{c}{|x|^d}, \qquad \text{for some } c > 0 \text{ and all } |x| \geq 1.$$

Conclude that f^* is not integrable on $\mathbb{R}^d$. Then, show that the weak type estimate

$$m(\{x : f^*(x) < \alpha\}) \leq \frac{c}{\alpha}$$

for all $\alpha > 0$ whenever $\int |f| = 1$, is best possible in the following sense: if f is supported in the unit ball with $\int |f| = 1$, then

$$m(\{x : f^*(x) > \alpha\}) \geq \frac{c'}{\alpha}$$

for some $c' > 0$ and all sufficiently small α.

Proof. f is not identically zero means there is some set E of nonzero measure such that $f \neq 0$ on E. Let C be a ball that contains E- without loss of generality we can suppose that C has radius 1 and is centered at the origin (the invariance properties of the Lebesgue integral allow us to perform this). $\displaystyle\int_C |f(y)|\ dy > 0$ because on E we have $f \neq 0$ so $|f| > 0$; f is integrable so we can call $\displaystyle\int_C |f(y)|\ dy = c_1, \infty > c_1 > 0$. By the property of the supremum, $f^*(x) \geq \dfrac{1}{m(B)} \displaystyle\int_B |f(y)|\ dy$ where we say B has radius x (so B can contain the point x), B centered at the origin. We know that the integral part is at least c_1 for $|x| \geq 1$ because $B \supseteq C$ in that case (monotonicity with the nonnegative function $|f(y)|$). Additionally from chapter 1 that the measure of B is $c_2 |x|^d$ where c_2 is some positive constant related to the volume of the ball (like $\dfrac{4\pi}{3}$ for the sphere). Thus, $f^*(x) \geq \dfrac{c}{|x|^d}$ with $\frac{c_1}{c_2} = c > 0$ and $|x| \geq 1$.

$f^*(x)$ is not integrable- by the above result we have that f^* is nonnegative and bounded below by $\dfrac{c}{|x|^d} > 0$ (since $c > 0$). Thus we demonstrate f^* is not integrable by noting that $\dfrac{c}{|x|^d}$ is not integrable. This is because the integral $\displaystyle\int_{\mathbb{R}^d} \frac{1}{|x|^d} = 2a \log|x|$ for some nonzero constant a (taking successive antiderivatives on the positive parts and negative parts respectively). Evaluating this integral over $\mathbb{R}^d$ we note that it does not converge. Thus, since f^* is bounded below by a nonintegrable function, f^* must be nonintegrable.

For the weak type estimate, let E_α be the set $\{x : f^*(x) > \alpha\}$; so, for $x \in E_\alpha$, there exists a ball B_x such that $x \in B_x$ and $\dfrac{1}{m(B_x)} \displaystyle\int_{B_x} |f(y)|\ dy > \alpha$. Now, notice that if we only are concerned with small values of α that the unit ball is always such a ball that contains x- since $\displaystyle\int |f| = 1$ on the unit circle, and f^* is the supremum over all balls containing x. Thus the measure of E_α is guaranteed to be the volume of the unit ball; thus we can choose c to be the appropriate constant such that $m(\{x : f^*(x) > \alpha\}) = 1 = \frac{c}{\alpha}$. ∎

Problem 5. Consider the function on $\mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{1}{|x|(\log 1/|x|)^2} & \text{if } |x| \leq \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

(a) Verify that f is integrable.

(b) Establish the inequality

$$f^*(x) \geq \frac{c}{|x|\,(\log 1/|x|)} \qquad \text{for some } c > 0 \text{ and all } |x| \leq 1/2,$$

to conclude that the maximal function f^* is not locally integrable.

(a) First, notice that $\int_{\mathbb{R}} f$ is really $\int_{\frac{-1}{2}}^{\frac{1}{2}} f$; from there, we can eliminate the absolute value symbols (f is an even function) by decomposing the integral into

$$\int_{\mathbb{R}} f = \int_0^{\frac{1}{2}} \frac{1}{x\left(\log\left(\frac{1}{x}\right)\right)^2}\, dx + \int_{\frac{-1}{2}}^0 \frac{1}{-x\left(\log\left(\frac{1}{-x}\right)\right)^2}\, dx$$

For the positive side, the u-substitution $u = \log\left(\frac{1}{x}\right)$, $du = -\frac{1}{x}dx$ gives us

$$\begin{aligned} \int_0^{\frac{1}{2}} \frac{1}{x\left(\log\left(\frac{1}{x}\right)\right)^2} dx &= -\int \frac{1}{u^2} du \\ &= \frac{1}{u} \\ &= \frac{1}{\log\left(\frac{1}{x}\right)} \end{aligned}$$

Evaluating the antiderivative at 0 and $\frac{1}{2}$ we see that this integral exists. Similarly, for the negative side (x is a negative value for what follows) use the u-substitution $u = \log\left(\frac{-1}{x}\right)$, $du = \frac{-1}{x}dx$

$$\begin{aligned} \int_{\frac{-1}{2}}^0 \frac{1}{-x\left(\log\left(\frac{1}{-x}\right)\right)^2} dx &= \int \frac{1}{u^2} du \\ &= \frac{-1}{u} \\ &= \frac{-1}{\log\left(\frac{-1}{x}\right)} \end{aligned}$$

Evaluating this antiderivative at $\dfrac{-1}{2}$ and 0 shows this integral also exists. Hence the integral $\displaystyle\int_R f$ is finite; f is integrable.

(b) Note that $f(x)$ as defined in the problem is already a nonnegative function. Using our results for the antiderivative, we have that for any ball B centered at the origin with radius r that $\displaystyle\int_B |f(y)|\,dy = \int_0^{|r|} |f(y)|\,dy + \int_{-|r|}^0 |f(y)|\,dy = \frac{1}{\log\left(\frac{1}{|r|}\right)} - 0 - \left(0 - \frac{-1}{\log\left(\frac{1}{|r|}\right)}\right) = \frac{2}{\log\left(\frac{1}{|r|}\right)}$. Thus, using the property of supremum, $\displaystyle f^*(x) \geq \frac{1}{m(B)} \int_B |f(y)|\,dy$; where B is a ball that contains x; in other words, the radius of B is $|x|$, so $\displaystyle\int_B |f(y)|\,dy = \frac{2}{\log\left(\frac{1}{|x|}\right)}$ and $m(B) = 2|x|c$ ($c > 0$ is a constant related to the volume of this ball). So $\displaystyle f^*(x) \geq \frac{c^{-1}}{|x|\log\left(\frac{1}{|x|}\right)}$; since c is just some positive constant, we can re-dress this result to get $\displaystyle f^*(x) \geq \frac{c}{|x|\log\left(\frac{1}{|x|}\right)}$ as desired.

With our inequality from above, we have that f^* is bounded below on the interval $(\frac{-1}{2}, \frac{1}{2})$ by the function $\dfrac{c}{|x|(\log 1/|x|)}$; thus, we can show f^* is not locally integrable by determining $\displaystyle\int_{\frac{-1}{2}}^{\frac{1}{2}} \frac{c}{|x|(\log 1/|x|)}$. Consider the positive part; this function is even, so we observe that our answer is twice the integral over just the positive half. Use the u-substitution $u = \log\dfrac{1}{x}$; $du = \dfrac{-1}{x}dx$

$$\begin{aligned}
\int_{\frac{-1}{2}}^{\frac{1}{2}} \frac{c}{|x|\log 1/|x|}\,dx &= 2\int_0^{\frac{1}{2}} \frac{c}{x\log 1/x}\,dx \\
&= -2\int_\infty^{\log 2} \frac{1}{u}\,du \\
&= 2\int_{\log 2}^\infty \frac{1}{u}\,du
\end{aligned}$$

By observing that the harmonic series diverges, by comparison this integral also diverges; thus the lower bound of f^* is not integrable, and hence f^* is not integrable either.