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quantum // mechanics

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1 Introduction

1.1 Entanglement, Locality, and Hidden Variables

The observation of various, tangible effects explained by quantum mechanics poses a question of interpretation- made famous in the Bohr-Einstein debates about the philosophical problems raised by quantum theory. For example, quantum mechanics posits that physical properties of a particle such as its position or momentum are at first undetermined, given only by a "probability wave function" that describes the various possibilities for a measured quantity (such as position) and the probability of finding the particle in any given possible state. Additionally, quantum mechanics allows for the possibility of multiple particles to be encoded by a single probability function- that is to say, it is possible to create a pair (or more) of particles such that it is impossible to only determine a physical property like position for one, without additionally determining it for the other. The term for this phenomenon is **quantum entanglement**.

This conclusion yields some very non-intuitive implications. It means that, if we had a procedure to create pairs of particles demonstrating this behavior, we could separate the two particles by a vast distance and then perform a measurement on one- in an instant, with no "transmission" of effects, we would also know what the property of the paired particle is. This is Einstein's "spooky action at a distance", and this is the heart of what is now termed **quantum nonlocality**. In short, *how did the other particle know what to do?*

In the classical view, **locality** is the principle that any object will only be influenced by its immediate surroundings. The conclusions of special relativity allow us to formulate this in a more rigorous sense: no material, energy, or even information can travel faster than the speed of light. The biggest proponent of locality as an axiom for the universe was Einstein, and indeed General Relativity was born out of asking how to explain gravity while maintaining this principle. Not to get too distracted, but the thought experiment behind it goes as follows: if the sun were completely removed from existence right now, would the Earth fly away even as light was still traveling towards us from the now-removed sun? What's the "speed" of gravity?

Thus, when the theoretical conclusions of quantum mechanics seem to present a challenge to locality, Einstein's resolution was to propose that quantum mechanics was as-yet incomplete; that there were still unknown **hidden variables** that were at play causing the entangled particles to have their measurements show up as being seemingly correlated with one another. As it turns out, it is impossible for the presence of hidden variables to explain quantum nonlocality.

1.2 Primer on Polarization

One of the particles that have been found capable of becoming entangled is the photon; through the use of various physical apparatus it is possible to create pairs of photons that are entangled in their polarization. Quantum mechanics asserts that an individual photon can be polarized in a similar way as light. The next section concerns a summary of how light polarization is described in a mathematical sense.

Light as an electromagnetic wave can be polarized. It is important to recognize that light is a wave traveling through three dimensional space, so its polarization is of a two-dimensional nature (the third dimension is the direction it's going in). Essentially, we can describe a light wave by combining two different waves, one in the x component and one in the perpendicular y component (again, the z component is aligned with the direction the light is going). Indeed, the simplest light polarization is linear polarization- light can be polarized so that it only oscillates in the x direction, and hence resembles a traditional sine wave. Similarly, we can have linear polarization in the y direction.

Finally, we can have the oscillations in the x and y direction line up as to have the wave spiral around like threads on a screw, either in the right-hand or left-hand direction.

To cut down the amount of three-dimensional mathematics required to describe these various possible ways for light to be polarized, there is a shorthand for describing waves. Any wave can be described using the symbols $|...\rangle$, and waves can be added and subtracted to each other as well as be scaled by a scalar constant.

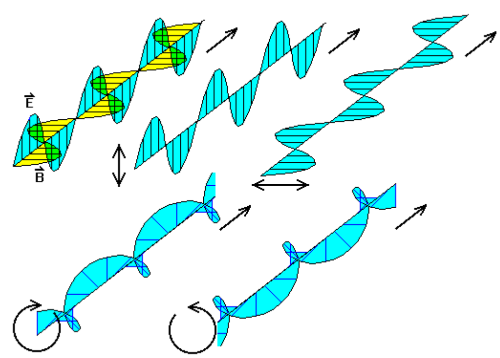

Figure 1: Various types of polarized light.

The following list is of the specific types of polarization we will give a name to:

- The wave for x-linear polarization is given the symbol $|h\rangle$, and the wave for y-linear polarization is given the symbol $|v\rangle$, in an analogy to something being "horizontally" or "vertically" polarized.

Note that our choice of x and y is made only on the principle that the x and y axes were perpendicular; in fact, light can be polarized linearly but not align with either the x or y axis.

- For future reference, the symbol for light that is linearly polarized in a 45° to the x axis (so, halfway between the x and y axis going counterclockwise) is called "diagonally" polarized and is given the symbol $|d\rangle$; similarly, light that is linearly polarized in a $-45°$ to the x axis (so, halfway between the x and y axis in a clockwise direction) is called "anti-diagonally" polarized and is given the symbol $|a\rangle$.

It is a routine right-triangle trigonometry exercise to notice that, for example, $|d\rangle = \frac{1}{\sqrt{2}}\big(|h\rangle + |v\rangle\big)$. Similarly, $|a\rangle = \frac{1}{\sqrt{2}}\big(|h\rangle - |v\rangle\big)$.

- Finally, to describe polarization turning in the "right-hand" direction (from the point of view of the source of light) we have $|r\rangle = \frac{1}{\sqrt{2}}\big(|h\rangle - i|v\rangle\big)$ and for "left-hand" polarization we have $|l\rangle = \frac{1}{\sqrt{2}}\big(|h\rangle + i|v\rangle\big)$.

The presence of the imaginary unit i is a matter of mathematical formalism to encode the clockwise and counterclockwise turning behavior.[1]

Quantum mechanical theory asserts that when we consider single photons prior to any measurement a photon has its possible polarization described only in a probabilistic sense. However, the difference is that now the states are undetermined; so when a photon polarization state (as distinct from a light wave polarization state) is given as $|\psi\rangle = \frac{1}{\sqrt{2}}\big(|h\rangle + |v\rangle\big)$ we are now discussing probabilities of finding the photon in either state $|h\rangle$ or $|v\rangle$. Given in this form, we state that the photon is in **superposition** between the polarization states $|h\rangle$ and $|v\rangle$; once a measurement is made, the photon will be found to be in either $|h\rangle$ or $|v\rangle$. Notice that all of the polarization possibilities listed above amount to an expression made up only of $|h\rangle$, $|v\rangle$,

[1]http://acko.net/blog/how-to-fold-a-julia-fractal for more information.

and constant multiplication by a real or complex number. Thus the photon, prior to measurement, is given the undetermined polarization state $|\psi\rangle = \alpha|h\rangle + \beta|v\rangle$, where α and β are complex constants such that $|\alpha|^2 + |\beta|^2 = 1$. The quantity $|\alpha|^2$ in this equation represents the probability of finding the photon in polarization state $|h\rangle$; similarly for $|\beta|^2$. This notion, of being able to describe any polarization in terms of a set of "base" states of polarization, will be important.

Finally, entangled photons can be described using subscripts. A non-entangled system of two photons can be described as $|\psi\rangle_1 = \alpha|h\rangle_1 + \beta|v\rangle_1, |\psi\rangle_2 = \gamma|h\rangle_2 + \delta|v\rangle_2$ where $|\alpha|^2 + |\beta|^2 = 1, |\gamma|^2 + |\delta|^2 = 1$. An entangled system appears when it is impossible to write the functions $|\psi\rangle_1, |\psi\rangle_2$ as two separate equations; for example, if we entangled two photons so that they always had the same polarization state then their probability wave equation will be $|\psi\rangle_{12} = |\psi\rangle_1|\psi\rangle_2 = \frac{1}{\sqrt{2}}\big(|h\rangle_1|h\rangle_2 + |v\rangle_1|v\rangle_2\big)$. Any entanglement equation describes a particular entanglement phenomenon; for example, using the above equation if photon 1 is measured and has $|h\rangle$ polarization, then what changes in the equation is that the state $|v\rangle_1$ is now known to be impossible; as a result, the probability of it occurring drops to zero, and thus $|v\rangle_1|v\rangle_2 = 0$; it does not matter that we have not measured $|v\rangle_2$, since it will never be the case that $|v\rangle_1$ the whole term drops out. So we are left with $|\psi\rangle_{12} = |h\rangle_1|\psi\rangle_2 = |h\rangle_1|h\rangle_2$; the conclusion is that $|\psi\rangle_2 = |h\rangle_2$, which is precisely what we know to be the phenomenon of entanglement.

1.3 Thought Experiment & Theorem

The following thought experiment and theorem establish the insufficiency of hidden variables as an explanation for quantum nonlocality.

The scenario is there is a central space station that first creates three entangled photons with entangled state $|\psi\rangle_{123} = \frac{1}{\sqrt{2}}\big(|h\rangle_1|h\rangle_2|h\rangle_3 + |v\rangle_1|v\rangle_2|v\rangle_3\big)$. These three photons are each sent to three separate observers, who themselves are all equipped with the same measurement apparatus. The measurement apparatus can either choose to measure the photon's polarization in terms of:

1. Right-hand versus left-hand polarization, or,
2. Diagonal versus anti-diagonal polarization.

For right-hand versus left-hand, the machine gives a "+1" for right-hand polarization being registered and a "−1" for left-hand. For diagonal versus anti-diagonal, the machine gives a "+1" for diagonal and a "−1" for anti-diagonal.

The observers each receive their photon, select what their machine will measure, and then record it. Afterwards, all three observers meet in the same place and compare their measurements. In particular, they compute the numerical product of their measurements.

Theorem 1.1. *Hidden variables fail to describe quantum nonlocality.*

Proof. Assume for sake of contradiction that there are hidden variables at play. That is to say, when three entangled photons are created and sent to the observers, some portion of their creation process "locks in" or otherwise describes the polarization of the entangled photons- in the strongest case, it could be that the photons always have a "true" polarization value that is simply out of reach from human observation due to uncertainty, but is still a part of the universe. Whatever the nature of the hidden variables, consider the following two scenarios:

1. Only one person uses the diagonal vs. anti-diagonal detector.

2. All three people use the diagonal vs. anti-diagonal detector.

The hypothesis that there is a hidden variable means that however these three entangled photons are entangled (the "true" polarization behind the entangled triplet might be that one must be vertical, one must be horizontal, and one must be right-hand polarized, for example), that it should remain consistent despite a change in measurement basis. We will demonstrate that a change in measurement basis in fact influences the possible entanglement outcomes, which contradicts the notion that whatever the "true" polarization outcomes are, they are independent of the measuring tool used. Hence, by choosing a measurement apparatus, one is indirectly choosing what quantity they will record by measuring; there *cannot* be a "true" value that stands independent of observers.

2 Mathematical Interlude

What follows is the exact derivation of the state function $|\psi\rangle_{123}$ for the two scenarios. At the abstract level, each one of the entangled photons comes from the Hilbert space $\mathbb{C}^2$ (we need two complex numbers, like how we needed α and β above, or how we described the polarization of light as a two-dimensional property); the space of particle entanglement outcomes is thus the tensor product of the respective Hilbert spaces each one of the particles comes from, and so the three-particle entanglement is thus an element of $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. The existence of quantum entanglement is the exact same as the existence of non-simple tensors, and what we are about to show with equations is that a change of basis operates unpredictably on non-simple tensors.

The algebra of tensors is straightforward. By and large it shares the same features as algebra on numbers with the exception of commutativity. Primarily, we will use the following rules:

- $\big(|X\rangle + |Y\rangle\big)\big(|A\rangle + |B\rangle\big) = |X\rangle|A\rangle + |Y\rangle|A\rangle + |X\rangle|B\rangle + |Y\rangle|B\rangle$
- $\alpha|X\rangle + \alpha|Y\rangle = \alpha\big(|X\rangle + |Y\rangle\big)$
- $|X\rangle|Y\rangle - |X\rangle|Y\rangle = 0$

The task will be to rewrite the expression $|\psi\rangle_{123} = \frac{1}{\sqrt{2}}\big(|h\rangle_1|h\rangle_2|h\rangle_3 + |v\rangle_1|v\rangle_2|v\rangle_3\big)$ in terms of the other bases we have defined; either using $|a\rangle, |d\rangle$ or $|r\rangle, |l\rangle$. The following rules are taken from section 1.2 and are manipulated into expressions of other base states:

- $|d\rangle = \frac{1}{\sqrt{2}}\big(|h\rangle + |v\rangle\big)$
- $|a\rangle = \frac{1}{\sqrt{2}}\big(|h\rangle - |v\rangle\big)$

$$\begin{aligned}\implies |d\rangle + |a\rangle &= \frac{1}{\sqrt{2}}2\big(|h\rangle\big)\\ &\implies \frac{1}{\sqrt{2}}\big(|d\rangle + |a\rangle\big) = |h\rangle\\ \implies |d\rangle - |a\rangle &= \frac{1}{\sqrt{2}}2\big(|v\rangle\big)\\ &\implies \frac{1}{\sqrt{2}}\big(|d\rangle - |a\rangle\big) = |v\rangle\end{aligned}$$

- $|r\rangle = \frac{1}{\sqrt{2}}\big(|h\rangle - i|v\rangle\big)$
- $|l\rangle = \frac{1}{\sqrt{2}}\big(|h\rangle + i|v\rangle\big)$

$$\implies |r\rangle + |l\rangle = \frac{1}{\sqrt{2}} 2\big(|h\rangle\big)$$

$$\implies \frac{1}{\sqrt{2}}\big(|r\rangle + |l\rangle\big) = |h\rangle$$

$$\implies |l\rangle - |r\rangle = \frac{1}{\sqrt{2}} 2i\big(|v\rangle\big)$$

$$\implies \frac{-i}{\sqrt{2}}\big(|l\rangle - |r\rangle\big) = |v\rangle$$

So, in conclusion: we have that $|h\rangle$ can be either $\frac{1}{\sqrt{2}}\big(|d\rangle + |a\rangle\big)$ or $\frac{1}{\sqrt{2}}\big(|r\rangle + |l\rangle\big)$, and that $|v\rangle$ can be either $\frac{1}{\sqrt{2}}\big(|d\rangle - |a\rangle\big)$ or $\frac{-i}{\sqrt{2}}\big(|l\rangle - |r\rangle\big)$.

2.1 Case 1

If only one observer is using the diagonal vs. anti-diagonal machine, without loss of generality let that person be labeled 1 (so their photon is labeled 1 as well). Now, we drop in the replacement rules we have in order to recast the $|h\rangle$ and $|v\rangle$ elements in terms of $|a\rangle$, $|d\rangle$, $|l\rangle$, and $|r\rangle$. In particular, since person 1 is using the diagonal vs. anti-diagonal machine, we want to turn $|h\rangle_1$ and $|v\rangle_1$ into expressions over the diagonal vs. anti-diagonal basis; for $|h\rangle_2$, $|h\rangle_3$, $|v\rangle_2$, and $|v\rangle_3$ we will switch to the right vs. left basis.

$$\begin{aligned}
|\psi\rangle_{123} &= \frac{1}{\sqrt{2}}\big(|h\rangle_1|h\rangle_2|h\rangle_3 + |v\rangle_1|v\rangle_2|v\rangle_3\big) \\
&= \frac{1}{\sqrt{2}}\Bigg(\left(\frac{1}{\sqrt{2}}(|d\rangle_1 + |a\rangle_1)\right)\left(\frac{1}{\sqrt{2}}(|r\rangle_2 + |l\rangle_2)\right)\left(\frac{1}{\sqrt{2}}(|r\rangle_3 + |l\rangle_3)\right) + \\
&\qquad \left(\frac{1}{\sqrt{2}}(|d\rangle_1 - |a\rangle_1)\right)\left(\frac{-i}{\sqrt{2}}(|l\rangle_2 - |r\rangle_2)\right)\left(\frac{-i}{\sqrt{2}}(|l\rangle_3 - |r\rangle_3)\right)\Bigg)
\end{aligned}$$

—monstrous amounts of multiplication and subsequent cancellation left to the reader—

$$|\psi\rangle_{123} = \frac{1}{2}\big(|a\rangle_1|r\rangle_2|r\rangle_3 + |d\rangle_1|r\rangle_2|l\rangle_3 + |d\rangle_1|l\rangle_2|r\rangle_3 + |a\rangle_1|l\rangle_2|l\rangle_3\big)$$

The takeaway from this is that the possible states for the entangled photons to be in after all three measurements are done are as follows:

- Anti-diagonal, and the other two observers record right-hand polarization.

- Anti-diagonal, and the other two observers both record left-hand polarization.
- Diagonal, and one observer records left-hand while the other observer records right-hand polarization.

Using the measurement apparatus described above, note that the product of their observations in any case will amount to -1.

2.2 Case 2

Now, we set up the situation where all observers are using the diagonal vs. anti-diagonal measurements. In a similar process to case 1, above:

$$\begin{aligned}
|\psi\rangle_{123} &= \frac{1}{\sqrt{2}}\big(|h\rangle_1|h\rangle_2|h\rangle_3 + |v\rangle_1|v\rangle_2|v\rangle_3\big) \\
&= \frac{1}{\sqrt{2}}\Bigg(\left(\frac{1}{\sqrt{2}}(|d\rangle_1 + |a\rangle_1)\right)\left(\frac{1}{\sqrt{2}}(|d\rangle_2 + |a\rangle_2)\right)\left(\frac{1}{\sqrt{2}}(|d\rangle_3 + |a\rangle_3)\right) + \\
&\qquad \left(\frac{1}{\sqrt{2}}(|d\rangle_1 - |a\rangle_1)\right)\left(\frac{1}{\sqrt{2}}(|d\rangle_2 - |a\rangle_2)\right)\left(\frac{1}{\sqrt{2}}(|d\rangle_3 - |a\rangle_3)\right)\Bigg)
\end{aligned}$$

—monstrous amounts of multiplication and subsequent cancellation left to the reader—

$$|\psi\rangle_{123} = \frac{1}{2}\left(|d\rangle_1|d\rangle_2|d\rangle_3 + |d\rangle_1|a\rangle_2|a\rangle_3 + |a\rangle_1|d\rangle_2|a\rangle_3 + |a\rangle_1|a\rangle_2|d\rangle_3\right)$$

Again, this means that the possible outcomes for the system after all measurements are finished are as follows:

- All three photons are recorded as diagonally polarized.
- One photon is recorded as diagonally polarized, and the other two are recorded as anti-diagonally polarized.

Using the measurement apparatus described above, this means the product of the three observations will always be 1.

To complete the proof, observe that $1 \neq -1$, the proof of which is too onerous to reprint here[2]. □

[2] Russell, B., 1910. '*Principia Mathematica*'

3 A Philosophical Coda

It is important to note that the phenomenon quantum mechanics seeks to describe is reproducible and exists outside of mathematical formalism. However, there is plenty of room for valid objection to theories and interpretations about these phenomenon. This theorem establishes that quantum entanglement is an example of something that the classical physics approach can never explain fully; there may still be hidden variables, and there may still be elements in current experimental physics that influence the validity of the experiments that purport to demonstrate non-local phenomenon.

However, if we accept the theorem, then we are left with an astounding choice of interpretations. For example, one could permit that phenomena can violate locality-such as with faster-than-light information transactions (which end up amounting to the implication that particles send information **back in time** to their entangled partners). Or, one could assert that it is indeed the case that physical properties of objects do not exist until they are “measured” in some sense. Taken to the extreme, this is either **solipsism** or a similarly exotic rejection of “reality” as a concept such as the many-worlds interpretation. Finally, one can still maintain locality and reality, but only through the admission of some form of **conspiracy**: at its weakest, that the entire universe in all of its existence is completely deterministic. This explanation is capable of explaining any phenomena, if only in a trivial sense.

Considering that denying reality or asserting conspiracy are both intellectual options no matter what phenomenon is under investigation, it should come as no surprise that the more popular interpretations of quantum mechanics and the insufficiency of hidden variables to explain non-locality is to assume that some minor degree of faster-than-light interaction, or “spooky action at a distance” is indeed possible.