1 2-D Heat Equation Stability

To analyze the stability conditions for the 2-D heat equation, observe that now there are two directions- the x direction and the y direction. As a result, we will now effectively be interpolating using the discrete Fourier transform in two directions: where there was in the past just the inner product $\sigma(f,g)$, there are now two inner products:

$$\sigma_x(f,g) = \sum_{0}^{M_x-1} f(x_i)g(\bar{x_i}), \sigma_y(f,g) = \sum_{0}^{M_y-1} f(y_i)g(\bar{y}+i) \tag{1}$$

Recall that a function defined only over a partition $\{x_0, x_1, ..., x_{M_x-1}\}$ is effectively a vector of the same length as the partition. As a result, we can interpret f and g as inhabiting vector spaces: V_x and V_y; this is just a rethinking of "function" over a discrete partition.

This suggests to us the proper way to make use of σ_x and σ_y: since our 2-D partition is in some sense parts x and parts y, we can think of the *function* partitioning x and y as inhabiting the vector space $V_x \otimes V_y(\{x_0, ..., x_{M_x-1}\} \times \{y_0, ..., y_{M_y-1}\})$; the inner product of this space will be the natural construction $\sigma_x \otimes \sigma_y$. The basis elements of this function will be all elements of the form $e^{ijx} \otimes e^{ikx}$.

The book contains the remainder of the proof of stability, using this as the foundation: it turns out to be in content the same as with the 1-D, with the tensor product here facilitating the notation.

2 Ch. 7: The Lax-Richtmyer Theorem

2.1 Premises

First we will need to clearly expand on the premises of this quite significant theorem. The theorem itself states that for any *consistent* finite difference method for a *well-posed* linear initial value problem that the method is consistent if and only if it is stable. In greater detail:

A *single-step* process is one in which we only make use of u^n to determine u^{n+1}- as we are used to in the setup $u^{n+1} = Au^n$.

For now, assume we are working only over 1 spatial dimension, with constant coefficients (this is noted to be a rather weak set of assumptions, which give away just how old this theorem is- contemporary problems in numerical analysis are much trickier).

Suppose that our solutions will come from a dense subspace V_0 of a Banach space V (this turns out to be a rather easy strengthening of the hypothesis- rather than assume our solutions will be drawn from a Banach space, we make use of the knowledge that every normed linear space *is* a dense subspace of a Banach space). Bounded operators on the dense subspace V_0 extend to V with the same norm (this is a common result- consult [Atkinson, Han]).

Our differential equation is a linear partial differential equation; that is, it has the following form:

$$\frac{du}{dt} = L(u); \qquad u : [0, T] \to V_0; \qquad u(0) = u_0 \tag{2}$$

Here, L are the spatial derivatives as a linear operator with constant coefficients (this is the typical statement of an initial value problem in differential equations).

Our problem is *well-posed* if the differential equation has a unique solution that varies continuously with initial values.

2.2 Exposition

A differential equation solution can be characterized by defining a map $S : [0, T] \to V_0$ where $S(t)$ is an operator on V_0 (so it has to have the property $u(t) = S(t)u_0$, and $S(t_1 + t_2) = S(t_1)S(t_2)$).

The finite difference method in particular gives rise to the operator R_k on V_0; for it to match the form above, we will interpret the single-step FDM as $R_k(u_0) = S(k)u_0$ and $R_k^m(u_0) = S(mk)u_0$- here R_k is taking the place of those state change matrices we have been working with.

The problem, and it's quite concrete, is that S is unknown. Indeed, S represents the solution to the problem- so as far as we are concerned, it is assumed to be out of our grasp (else, why would we be approximating the solution?). If we can define R_K such that $\|R_k\|$ is uniformly bounded and show convergence to S in the spatial partition then we will have shown the *Lax-Richtmyer Theorem*. For now, without showing it, we will show how this remarkable theorem is used.

2.3 Uses

We can bring this theorem to bear for any operator such that $\|R_k\| < c < \infty \forall k > N$, some N. In the situation with the FDM we have seen already that R_k is the exact same as the state change matrix $u^{n+1} = C_k u^n$; for the finite element method it will turn out to be the same as $I + kK = R_k$.