1 Infinite Vector Product & Direct Sum Isomorphism

1.1 The Isomorphism

Let's continue where we left off. We wish to show that

$$\left(\bigoplus_{i\in\Lambda} V_i\right)^* \cong \prod_{i\in\Lambda} V_i^*$$

The most ready way to do that is to find a map between the two spaces and show it is an isomorphism. What follows is a walkthrough of the deduction of the isomorphism. We will try to find the mapping from the right space to the left space. That is, let's start with a generic map $\varphi : \prod V_i^* \to \left(\bigoplus V_i\right)^*$. What do we know about these spaces?

By definition, $\left(\bigoplus V_i\right)^*$ is $\mathrm{Hom}\left(\left(\bigoplus_{i\in\Lambda} V_i\right), K\right)$ where K is the underlying scalar field for these vector spaces. Similarly, we know that $\prod V_i^*$ is by the bijective function correspondence the set of all functions $\left\{f : \Lambda \to \bigcup_{i\in\Lambda}^{\infty} V_i^*\right\}$; remember that V_i^* is itself the set of all functions in $\mathrm{Hom}\,(V_i, K)$. For clarity, from this point on we will adopt the following convention: elements of the direct sum will be f, g, h... and elements of the product will be F, G, H... The domain of φ contains elements that are functions of functions; so our map must be $\varphi(F)(f)$ where $f(i) \in V_i$ is almost everywhere zero.

Let me attempt to be more clear as to what F and f represent. f is a function such that $f(i) \in V_i$ for every index i and f is almost everywhere zero. Similarly $F(i) \in V_i^*$ is a function such that at every index i, $F(i)$ is an element of the dual- or, since the dual is the set of all linear transformations of V_i, for any index i the function $F(i)$ evaluates to a linear transformation of V_i. In other words, $F(i)\,(f(i)) \in K$, the underlying field (since $F(i)$ is a linear transformation of V_i to K, and $f(i)$ is some element of V_i).
Because of this, with a little creativity we conjecture that the isomorphism map we want is

$$\varphi(F)(f) = \sum_{i\in\Lambda} F(i)\,(f(i))$$

Remember that $\varphi(F)(f)$ is broken down as the function given by $\varphi(F)$ applied to the function f. The isomorphism arises because φ takes F to $\varphi(F)$; $F(i)$ is a function in $\mathrm{Hom}\,(V_i, K)$ and $\varphi(F)$ is a function in $\mathrm{Hom}\left(\bigoplus V_i, K\right)$.
Working with this conjecture, we need to show three facts to prove that φ is an isomorphism:

- φ is well-defined: that is, two different inputs should always give us two different outputs.

- φ is a homomorphism.

- φ is bijective.

To show φ is well-defined we must show that $\varphi(F)$ is a member of the set $\operatorname{Hom}\left(\bigoplus_{i\in\Lambda} V_i\right)^*$. That way, $\varphi(F)$ will unambiguously be what we want it to be. To show this fact we merely need to show $\varphi(F)$ is a homomorphism from $\bigoplus_{i\in\Lambda} V_i$ to K, since that is the definition of the set $\operatorname{Hom}\left(\bigoplus_{i\in\Lambda} V_i\right)^*$

Claim: $\varphi(F)$ is a homomorphism:

Proof.

$$\begin{aligned}\varphi(F)(f+g) &= \sum_{i\in\Lambda} F(i)(f+g)(i)\\ &= F(i)(f(i)+g(i))\end{aligned}$$

$F(i) \in V_i^*; f, g \in V_i$

$$\begin{aligned}&= \sum_{i\in\Lambda} F(i)f(i) + F(i)g(i)\\ &= \sum_{i\in\Lambda} F(i)f(i) + \sum_{i\in\Lambda} F(i)g(i)\\ &= \varphi(F)(f) + \varphi(F)(g)\end{aligned}$$

$\square$

Homework is to show the scalar case for $\varphi(F)$.
Next, we need to show φ is also a homomorphism:
Claim: φ is a homomorphism:

Proof.

$$\begin{aligned}\varphi(\alpha F)(f) &= \sum_{i\in\Lambda} (\alpha F)(i)(f(i))\\ &= \sum_{i\in\Lambda} \alpha F(i)(f(i))\\ &= \alpha \sum_{i\in\Lambda} F(i)(f(i))\\ &= \alpha\varphi(F)(f)\end{aligned}$$

$\square$

Homework is to show the addition case for φ.
Finally, we have to show φ is bijective. The way we will do this is show φ is a surjective and injective map.
Claim: φ is bijective:

Proof. Surjective case: Let $\mathcal{F} \in (\bigoplus V_i)^*$. Thus for any i, $\mathcal{F}(f(i)) \in K$; $\forall i, v_i \in V_i, \exists f \in \bigoplus_{i\in\Lambda} V_i$ given by $f(i) = v_i$. Hence $\mathcal{F}(f(i)) \in K$ is defined. Thus every element in the codomain is in the image (this is necessary, else we may have some elements $\mathcal{F}$ that are not in the target of the isomorphism!).
Now: $\forall j : \rho_j \left(\bigoplus V_i\right)^* = V_j^* \implies \rho_j(\mathcal{F}) \in V_j^*$. This tells us what the preimage of (F) should be: define $F \in \prod_{i\in\Lambda} V_i^*$ to be so $F(j) = \rho_j(\mathcal{F})$. Thus for any $\mathcal{F} \in (\bigoplus V_i)^*$ we claim $\varphi(F) = \mathcal{F}$.
Claim: $\varphi(F) = \mathcal{F}$

Proof.

$$\begin{aligned}\varphi(F)(f) &= \sum_{i\in\Lambda} F(j)f(j)\\ &= \sum_{i\in\Lambda} \rho_j(\mathcal{F})f(j)\\ &= \mathcal{F}(f)\end{aligned}$$

□

Hence $\varphi(F)(f) = \mathcal{F}(f)$, and so φ is surjective. □

For homework, show the injective case (it's easier than the surjective case).

2 More About Duals

2.1 Double Duals

Immediate results from our work so far: the dual space is "bigger" than the space: in the finite case $|V| = |V^*|$ and in the infinite case $|V^*| > |V|$ (just think about our existing isomorphisms). Consider also the dual-dual, $V^{**} = \mathrm{Hom}\,(V^*, K)$; for $x \in V, f \in V^*$ define $X^{**} \in V^{**}$ by $X^{**}(f) = f(x)$; the map $x \to X^{**}$ using $X^{**}(f) = f(x)$ is an isomorphism. Thus V^{**} and V are very closely related, much more than V^* and V. Sometimes these are referred to as "double dual" spaces.

2.2 Physical Examples

Dual spaces come up in particle physics. The classic example: a Hilbert space (such as R^n but for things like special relativity the space may be more exotic) is the space that contains particles, and the dual space to a given particle is what contains all the interactions (the self-adjoint operators) between the given particle and all the other particles in the space. Specifically, consider the photoelectric effect: electrons are the elements of the original Hilbert space, photons are what constitute the dual space. The double dual space will be the electron interactions that occur as a result of the photon interactions. In physical terms: photons act on electrons by raising their energy levels; electrons act on photons back via other mechanisms such as emission.

3 Symmetric Bilinear Forms

3.1 Definition

If V is a vector space we can define a *symmetric bilinear form* σ:

$$\sigma : V \times V \to K$$

$$\sigma(u, v) = \sigma(v, u)$$

$$\sigma(u + w, v) = \sigma(u, v) + \sigma(w, v)$$

$$\sigma(\alpha u, v) = \alpha\sigma(u, v)$$

For any vector $v \in V$ we can define the function $f_v(w) = \sigma(v, w)$ as a symmetric bilinear form: thus, since $f_v \in V^*$, the dual space gives rise to the context we need to talk about symmetric bilinear forms. If we think of the mapping $\varphi : v \to f_v$ then the kernel of this mapping is precisely the set $\{v \in V : \sigma(v, w) = 0 \forall w \in V\}$. This is the same as saying $v \in V^{\perp}$ (a symmetric bilinear form gives rise to the concept of orthogonality between vectors). Be aware that this condition seems strange to us for a reason: that is because our intuition will lead us to believe that nothing can be perpendicular (or equivalently, orthogonal) to everything. However, this is a condition that is not necessarily true in an arbitrary vector space: it is something we need to be explicit about.

3.1.1 An Example

If we take $\sigma(v, w) = \displaystyle\int_a^b Dv\ Dw\ dx$ where D is the differentiation operator and $v, w \in C^1[a, b]$. If $v = 0$ the zero function then $\sigma(v, w) = 0$ for all w. This is an example of something that can be orthogonal to everything in a space.

3.1.2 The Reisz Representation Theorem

This is what occurs with “nongeneracy”: a degenerate form σ is one whose kernel is nonempty, one whose subsequent definition of orthogonality gives rise to elements that are orthogonal to everything. One way of avoiding this is to guarantee that the form is “positive definite”: $\sigma(v, w) > 0 \forall v \neq \theta$. This turns the symmetric bilinear form into an inner product- what we call a *positive definite inner product*. As a final note, inner products of this nature give rise to a certain isomorphism theorem: the Riesz Representation Theorem.

Theorem 1. *(Riesz)*: In a Hilbert space H, for any function $f \in \mathrm{Hom}(H, \mathbb{R})$ there exists a unique vector $v \in H$ such that $f(w) = f_v(w) = \sigma(w, v)$