# 1 Matrix Representation

## 1.1 Definition

Let $V$ be a finite dimensional vector space, $B = \{v_1, ..., v_n\}$ a basis for it. For any vector $v \in V : \exists! \{\alpha_1, ., \alpha_n\}$ with $v = \sum_i^n \alpha_i v_i$. The alphas are unique because of the basis condition; we call the column vector $\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$ the *coordinatization* of $v$ at $B$, otherwise written as $[v]_B$.

Given any linear transformation $L \in \mathrm{Hom}(v)$ the *matrix representation* of $L$, $[L]_B$ is the matrix with columns $[L(v_i)]_B$ (the images of the basis vectors under $L$).

## 1.2 Uses of the Matrix Representation

With this in hand, we see that the operation of matrix multiplication works in the same way as a linear transformation does on vectors:

$$[L]_B [v]_B = [L(v)]_B$$

$$\begin{tikzcd} V \arrow[r, "L(v)"'] \arrow[d, "{[v]_B}"] & V \arrow[d, "{[L(v)]_B}"] \\ F^n \arrow[r, "{[L]_B}"'] & F^n \end{tikzcd}$$

Of course, this fact requires proof:

*Proof.* Let $v \in V$; by the basis we have $v = \sum_{i=1}^n \alpha_i v_i$. Consider $[L(v)]_B$:

$$\begin{aligned} [L(v)]_B &= \left[ L\left( \sum_{i=1}^n \alpha_i v_i \right) \right]_B \\ &= \left[ \sum \alpha_i L(v_i) \right]_B \\ &= \sum \alpha_i [L(v_i)]_B \end{aligned}$$

Coordinatization of linear combination is the coordinatization of the components

$$= \sum \alpha_i [L]_B^{(i)}$$

the $i$th column of the matrix $[L]_B$

$$= \sum \alpha_i [L]_B e_i$$

$e_i$ is the $i$ basis vector; this is a standard trick to pull columns out of a matrix

$$= \sum \alpha_i [L]_B [v_i]_B$$

the coordinatization of a basis vector of itself is $e_i : v_i = 0v_0 + 0v_1 + ... + 1v_i + ... + 0v_n$

$$\begin{aligned} &= [L]_B \left( \sum \alpha_i [v_i]_B \right) \\ &= [L]_B [v]_B \end{aligned}$$

coordinatization preserves linear combinations

$$= [L]_B [v]_B$$

$\square$

The matrix representation also yields an identity for composing linear transformations:

$$[L]_B [N]_B = [LN]_B$$

The process of composition of transformations is the same as the process of matrix multiplication; in essence, matrix multiplication is defined the way it is in order for this representation of composition to work.

# 2 Vector Products and Vector Direct Sums

## 2.1 Finite Product and Direct Sum

### 2.1.1 Vector Space

Let $V_1, ..., V_n$ be vector spaces; we know $\prod_{i=1}^{n} V_i$ by the name of its isomorphic space $\bigtimes_{i=1}^{n} V_i$, the Cartesian product of the spaces. This new space, along with the coordinate operations from $V_i$, yields a new vector space.

### 2.1.2 Projection and Injection Maps

The projection map $\rho_j : \prod_{i=1}^{n} V_i \to V_j$ is defined by stripping out only $V_j$; the injection map $\iota_j : V_j \to \prod_{i=1}^{n} V_i$ sends $V_j$ to $0, 0, ..., V_j, 0, 0$ (the $j$th spot is $V_j$, all the others are 0). Both $\rho_j$ and $\iota_j$ are linear maps.

### 2.1.3 Direct Sum

Let $V$ be a vector space, $V_1, V_2$ be subspaces. Define $V_1 + V_2 = \{v_1 + v_2 = v \in V\}$ be the subspace created by vectors that can be broken down into representations with component vectors in $V_1$ and $V_2$. The following theorem concerns such subspace breakdowns which have unique representation and the property that the subspaces must have; namely, that the intersection of the two subspaces is the zero vector iff vectors in $V$ have a unique breakdown by the two subspaces.

**Theorem 1.** $V_1 \bigcap V_2 = \{\theta\} \Leftrightarrow \forall v \in V_1 + V_2 : \exists! v \in V : v = v_1 + v_2$

*Proof.* $\Rightarrow$: Suppose for sake of contradiction that there is a nonzero vector without a unique representation: $v = v_1 + v_2 = \overline{v}_1 + \overline{v}_2$. This turns into $v = v_1 - \overline{v}_1 = \overline{v}_2 - v_2$; now we have that $v$ can be written entirely as vectors contained in $V_1$ and also as vectors in $V_2$. In other words, $v \in V_1 \bigcap V_2$; but by the hypothesis, this means $v$ must be the zero vector. Thus by contradiction, for any vector, there must be only a unique representation of that vector as the sum of vectors from $V_1$ and $V_2$.
$\Leftarrow$: Suppose for contradiction that there is a nonzero $v$ in the intersection $V_1 \bigcap V_2$. Starting with the hypothesis, every nonzero vector in $v$ has a unique representation as the sum of vectors from $V_1$ and $V_2$. Consider the following equation: $v = v + \theta = \theta + v$; these are two different ways to write $v$ as the sum of two vectors from $V_1$ and $V_2$, but by the hypothesis these must be the same representation- this can only be possible if $v = \theta$. Thus by contradiction $V_1 \bigcap V_2 = \theta$. □

When $V_1 \bigcap V_2 = \theta$ we can define $V_1 \times V_2$ as the *direct sum* $V_1 \bigoplus V_2$. With $n$ vector spaces we can generalize the direct sum; the condition generalizes into $V_i \bigcap V_j = \theta; i \neq j$ for every pair of vector spaces. The projection map is $\rho_j : \bigoplus_{i=1}^{n} V_i \to V_j$ and the injection map is $\iota_j : V_j \to \bigoplus_{i=1}^{n} V_i$. The direct sum notation is required because we require unique representations to have unambiguous projections and injection mappings. With the unique representation and unambiguous projections and injections, it turns out that the vector space given by the direct sum (along with the projection and injections for the direct sum) are isomorphic to the product vector space and its projections and injections.

$$\bigoplus_{i=1}^{n} V_i \cong \prod_{i=1}^{n} V_i$$

The last important tool is the bijection between a vector product space and the set of functions mapping $n$-tuples to the formal union $\bigcup_{i=1}^{n} V_i$

$$\prod_{i=1}^{n} V_i \cong \left\{ f : [1, ..., n] \to \bigcup_{i=1}^{n} V_i \right\}$$

In prose this is understood as a set of functions that have the following character: $f(i) \in V_i$ for $i \in [1, ..., n]$. For example, if we have $n$ vector spaces we can operate on two vectors taken from the product space:

$$(v_1, ..., v_n) + (w_1, ..., w_n) = (v_1 + w_1, ..., v_n + w_n)$$

By using the function definition of the vector product space we see that this is exactly the same under function addition and scalar multiplication as

$$\{(f + g)(i) = f(i) + g(i)\} : i \in [1, ..., n]$$

## 2.2 Countable Products

Generalizing one step further, consider the product $\prod_{i=1}^{\infty} V_i$. We would like to say that this is equivalent to a countable Cartesian product in the same way of the finite case- however, recall from topology that a countable Cartesian product could be uncountable. To get a better handle on this we can we should re-interpret the countable product: "the set of all sequences where the $i$th entry comes from the $i$th vector space"; coordinate operations now correspond to entry operations in sequences, and we express the sum of two vectors as

$$(v_1, v_2, ...) + (w_1, w_2, ...) = (v_1 + w_1, v_2 + w_2, ...)$$

The projection and injection maps still work,

$$\rho_j : \prod_{i=1}^{\infty} V_i \to V_j$$

$$\iota_j : V_j \to \prod_{i=1}^{\infty} V_i$$

Remember that these do not map whole spaces to each other, only elements inside the domain to elements inside the range. For instructive purposes, consider the image of the injection map $\iota_j V_j$ This is a subspace:

$$\iota_j V_j \leqslant \prod_{i=1}^{\infty} V_i$$

Consider the linear span of $\iota_j(v_j)$ for $j = 1, ..., \infty$. The span is isomorphic to all finite linear combinations; that is, the set of all sequences with finitely many nonzero entries. This artifice is necessary to wrap our heads around countable vector space products. This linear span is a proper vector subspace of $\prod_{i=1}^{\infty} V_i$; incidentally, this proper subspace is $\bigoplus_{i=1}^{\infty} V_i$; the proper is because while the direct sum cannot contain infinite sequences, the vector product clearly does contain infinite sequences. Observe how the direct sum is no longer isomorphic to the product as was in the finite case.
Note that the bijection to sets of functions works seamlessly from before:

$$\prod_{i=1}^{\infty} V_i \cong \left\{ f : \mathbb{N} \to \bigcup_{i=1}^{\infty} V_i \right\}$$

The direct sum subspace is therefore

$$\bigoplus_{i=1}^{\infty} V_i = \left\{ f : \mathbb{N} \to \bigcup_{i=1}^{\infty} V_i \right\}$$

where the function $f$ are almost always zero (that is, they are nonzero on finitely many indicies).

## 2.3 Uncountable Products

The next level of generalization is to uncountable vector space products. Observe that already we have difficulty in the form of *writing down* what we mean by an uncountable vector product. In this case we have some indexing set $\Lambda$; the product is thus $\prod\limits_{i\in\Lambda} V_i$

Here the only tool that gives us any headway on this concept is by using the artifice of the bijection to sets of functions. In this case,

$$\prod_{i\in\Lambda} V_i \cong \left\{ f : \Lambda \to \bigcup_{i\in\Lambda}^{\infty} V_i \right\}$$

where $f(i) \in V_i$. Vector space operations are thought of in the same way as addition and scalar multiplication of functions.

The injection $\iota_j\left(V_j\right) = \left\{ f \in \prod\limits_{i\in\Lambda} V_i : f(i) = 0 \text{ if } i \neq j, f(j) = \text{something in } V_j \right\}$. It injects into the $j$th component in the product space. The projection $\rho_j\left(\prod\limits_{i\in\Lambda} V_i\right) = V_j$ works the same way; it isolates out the $V_j$ vector space; with the function notation, $\rho_j : \prod_{i\in\Lambda} V_i \to V_j$; $\rho_j(f) = f(j)$ where $f$ is from the function notation.

The direct sum of the uncountable vector space collection is $\bigoplus\limits_{i\in\Lambda} V_i = \left\{ f \in \prod\limits_{i\in\Lambda} V_i \right\}$ : where $f(i)$ almost always zero.

# 3 Dual Spaces

Now that we are familiar with product spaces in the finite, countable, and uncountable cases, we turn our attention to *dual spaces* of vector spaces. Given a vector space $V$ the *dual vector space* $V^*$ of $V$ is the set of all linear transformations $\mathrm{Hom}(V, F)$ where $F$ is the field of scalars of the vector space (remember a vector is just a collection of scalars defined over some underlying field).

## 3.1 Finite Case

Let $V$ be a finite dimensional vector space, $B$ a basis $(v_1, ..., v_n)$; define $f_n : V \to F$ such that $f_n(v_m) = \delta_{n,m}$ where $\delta_{n,m}$ is the Kronecker delta (if $n = m$ $\delta_{n,m} = 1$, else $\delta_{n,m} = 0$).

The basis of the original vector space is directly related to the basis of the dual space:

**Theorem 2.** $\{f_1, ..., f_n\}$ is a basis for $V^*$.

*Proof.* The number of elements in this set is the correct amount for a basis, so to show it is a basis (via the maximally independent property) we want to show that the set in question is linearly independent. Let $\sum \alpha_i f_i = 0$ be a linear combination; observe that the zero here is the zero function, not the zero scalar. Operate this linear combination on any basis vector $v_j$: $0 = \sum \alpha_i f_i(v_j) = \alpha_j$ (since the Kronecker delta annihilates all but the $j$th index). Since this is true for any $j$, we have that all the $\alpha_j$ must be zero; this is the trivial linear combination. Hence $\{f_1, ..., f_n\}$ is a basis for $V^*$. □

## 3.2 Uncountable Case

This is where things get difficult.

Given an uncountable vector space $V$ over field $F$ with basis $B = v_i : i \in \Lambda$ (guaranteed to exist by a previous theorem) we have that this specific basis gives us the collection of functions $f$ such that $f : \Lambda \to F; f_i = f(i)$ is zero almost everywhere and $\sum_{i \in \Lambda} f_i v_i$ is a vector in $V$; the dual space is therefore established by taking all the functions from $\Lambda$ to $F$ (this bit should remind you of the artifice we used in uncountable vector space products: the dual of a given $V_i$ is the set of all linear transformations $V_i$ to $F$ and so $\prod_{i \in \Lambda} V_i = \left\{ f : \Lambda \to \bigcup^{\infty} V_i \right\}$ with $f(i) \in V_i$). Recall that the uncountable direct sum is isomorphic to the set of all functions $f \in \prod_{i \in \Lambda} V_i : f(i)$ almost always zero; the dual space, is isomorphic to all possible sequences of linear transformations.

The relation we have been pushing for is, in other words:

$$\left( \bigoplus_{i \in \Lambda} V_i \right)^* \cong \prod_{i \in \Lambda} (V_i^*)$$

The last, extremely challenging and unanswered question, is what the isomorphism map is between these two objects. We have shown that they are the same thing, but we have not produced any possible isomorphism; it turns out to be very difficult, but in the scope of this class, to find it.