1. Newton's Method (25)

   (a) Derive the operative statement for Newton's method of finding roots of $f(x)$ with the initial guess of x_0.

   (b) Apply Newton's Method to the function $f(x) = (x-1)(x-2)$ and guess $x_0 = 0$ once.

   (c) Provide three statements that explain the advantages or disadvantages of Newton's method over the secant method.

2. Linear Systems (25)

   (a) Define the operator norm of a matrix, $\|A\|$.

   (b) Define the condition number for A.

   (c) Given $\|Av\| \leq \|A\|\|v\|$, prove $|\alpha| \leq \|A\|$ given α is an eigenvalue of A.

   (d) (EC) Prove $\|Av\| \leq \|A\|\|v\|$.

3. Minima Finding (25)

   (a) How can we use a minima finding procedure to find roots?

   (b) Let $f(x, y) = x^2 + y^2$ and let $(1, 2)$ be the start point. Execute the next iteration of the gradient descent line search (final answer should be of the form (x, y), a new point).

   (c) State at least one reason a minima finding algorithm can perform poorly.

4. Interpolation (25)

   (a) Write down the Vandermonde matrix V for interpolating $(x_0, y_0), (x_1, y_1), (x_2, y_2)$.

   (b) Given data points (x_i, y_i) and the $n+1$ Lagrange basis polynomials $l_i(x)$, state the interpolating polynomial $p(x)$ as a linear combination of Lagrange basis polynomials.

   (c) Provide three statements that explain the advantages or disadvantages of Vandermonde matrix interpolation over Lagrange basis polynomial interpolation.

5. Topic (10): Recall that *ill-conditioned* means that a small change in the input leads to a large change in the value of the output. Similarly, *well conditioned* means that a small change in the input will lead only to a small change in the output.

   (a) Given polynomial $p(x) = \sum a_n x^n$: is determining the roots by an algorithm that operates only on coefficients (such as factorization, or the quadratic formula) in general a well-conditioned problem? (hint: consider finding the roots of $x^2 - \epsilon$, where ϵ is the coefficient to be changed).

   (b) (*Wilkinson's polynomial, 1963*) Let $w(x) = \prod_{i=1}^{20}(x-i) = (x-1)(x-2)...(x-20)$ be a polynomial with roots $(1, 2, ..., 20)$. Let $c(x)$ be another polynomial of degree 20 and thus for some perturbation factor t let $w(x) + t \cdot c(x)$ be the polynomial $w(x)$ under a small change in coefficients. Thus for a given root α_i the derivative $\dfrac{\partial \alpha_i}{\partial t} = -\dfrac{c(\alpha_i)}{w'(\alpha_i)}$ is the **change in the root given a change in** t. Let $c(x) = x^{19}$ (so we are changing the coefficient of the x^{19} term by t); this derivative is therefore $\dfrac{\partial \alpha_i}{\partial t} = -\dfrac{\alpha_i^{19}}{\prod_{k \neq i}(\alpha_i - \alpha_k)} = -\prod_{k \neq i} \dfrac{\alpha_i}{\alpha_i - \alpha_k}$. From this expression, what can you say about the stability of the roots of $w(x)$? Is 1 a stable root? What about 20?

   (c) One procedure for finding the eigenvalues of a matrix A is to find the characteristic polynomial and solve for the roots of the characteristic polynomial. Using what you have just read, is this procedure numerically stable for well-conditioned matrices? Why or why not?

Speaking for myself I regard [w(x)] as the most traumatic experience in my career as a numerical analyst - James H. Wilkinson, 1984

1. Newton's Method (25)
   (a) Derive the operative statement for the secant method of finding roots.
   (b) Provide one example of a situation where Newton's method will fail to work.
   (c) Provide three statements that explain the advantages or disadvantages of Newton's method over the secant method.
2. Linear Systems (25)
   (a) Define the operator norm of a matrix, $\|A\|$.
   (b) Given a linear system $A\vec{x} = \vec{b}$ and $\vec{e} = \vec{x} - \hat{x}$ and $\vec{r} = \vec{b} - \hat{b}$, explain what is meant by the following statement: $\frac{\|\vec{e}\|}{\|\vec{x}\|} \leq \frac{\|A\|\|A^-1\|\|\vec{r}\|}{\|\vec{b}\|}$.
   (c) If α is an eigenvalue of A, prove that $\frac{1}{\alpha}$ is an eigenvalue for A^{-1}.
   (d) (EC) Prove that $\frac{\|\vec{e}\|}{\|\vec{x}\|} \leq \|A\|\|A^{-1}\|\frac{\|\vec{r}\|}{\|\vec{b}\|}$ where $A\vec{x} = \vec{b}$ and $\|\vec{e}\| = \|\vec{x} - \hat{x}\|$ and $\|\vec{r}\| = \|\vec{b} - \hat{b}\|$ with $\hat{x}$ the computed solution and $\hat{b} = A\hat{x}$.
3. Minima Finding (25)
   (a) How do we use a minima finding procedure on vector-valued functions (like a vector field)?
   (b) Let $f(x, y) = x^2 + y^2$ and let the initial guess be $(1, 2)$. Using the Newton's method line search (the one with the Hessian matrix), step through one iteration of the procedure.
   (c) State at least one reason a minima finding algorithm can perform poorly.
4. Interpolation (25)
   (a) Prove that the Vandermonde matrix polynomial interpolation is the same as the Lagrange basis polynomial interpolation.
   (b) Given the domain $[a, b]$ and the partition $\{a = x_0, x_1, x_2 = b\}$, define the three Lagrange basis polynomials of degree 2.

5. Topic (10): Recall that *ill-conditioned* means that a small change in the input leads to a large change in the value of the output. Similarly, *well conditioned* means that a small change in the input will lead only to a small change in the output.

   (a) Given polynomial $p(x) = \sum a_n x^n$: is determining the roots by an algorithm that operates only on coefficients (such as factorization, or the quadratic formula) in general a well-conditioned problem? (hint: consider finding the roots of $x^2 - \epsilon$, where ϵ is the coefficient to be changed).

   (b) (*Wilkinson's polynomial, 1963*) Let $w(x) = \prod_{i=1}^{20}(x-i) = (x-1)(x-2)...(x-20)$ be a polynomial with roots $(1, 2, ..., 20)$. Let $c(x)$ be another polynomial of degree 20 and thus for some perturbation factor t let $w(x) + t \cdot c(x)$ be the polynomial $w(x)$ under a small change in coefficients. Thus for a given root α_i the derivative $\frac{\partial \alpha_i}{\partial t} = -\frac{c(\alpha_i)}{w'(\alpha_i)}$ is the **change in the root given a change in** t. Let $c(x) = x^{19}$ (so we are changing the coefficient of the x^{19} term by t); this derivative is therefore $\frac{\partial \alpha_i}{\partial t} = -\frac{\alpha_i^{19}}{\prod_{k \neq i}(\alpha_i - \alpha_k)} = -\prod_{k \neq i} \frac{\alpha_i}{\alpha_i - \alpha_k}$. From this expression, what can you say about the stability of the roots of $w(x)$? Is 1 a stable root? What about 20?

   (c) A polynomial in this form $p(x) = \sum_{i=0}^{n} a_i x^i$ expresses the polynomial over a particular basis: $\{1, x, x^2, ..., x^n\}$. In $w(x)$, above, we have that $w(x) = \sum_{i=1}^{20} a_i x^i$. Using the Lagrange basis polynomials $l_k(x) = \prod_{i \in \{0,...,20\} \setminus \{k\}} \frac{(x-i)}{(k-i)}$, rewrite the polynomial as $w(x)$: $w(x) = \sum_{k=0}^{20} d_k l_k(x)$ (i.e. solve for the d_k).

   (d) Given $w(x) = \sum_{k=0}^{20} d_k l_k(x)$ over the Lagrange basis, how does a change in the coefficients d_k change the roots α_k? Is $w(x)$ well conditioned or ill-conditioned in this basis?

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