Fundamental Theorem of Algebra

1 Theorem & Proof

We will need the following result of Eugène Rouché, presented without proof:

Rouché's Theorem: *If f and g are complex-valued functions holomorphic inside and on a simple closed contour K with $|g(z)| < |f(z)|$ on K, then f and $f+g$ must have the same number of zeroes inside K (counting multiplicities).*

Fundamental Theorem of Algebra: *Every non-zero polynomial with complex coefficients $p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ has exactly n complex roots (counting multiplicities).*

Proof. Given $p(x)$, above, choose a number $R > 0$ large enough that

$$|a_{n-1}x^{n-1} + ... + a_1 x + a_0| \leq \sum_{j=0}^{n-1} |a_j| R^{n-1} < |a_n| R^n = |a_n x^n|$$

for all $|x| = R$ (x may be a complex number; in this treatment we will assume that x is only real-valued).
Since $a_n x^n$ has n zeroes (counting multiplicities) inside the disk $|x| < R$ (since $R > 0$), Rouché's theorem that $p(x)$ must also have n zeroes (counting multiplicities) inside the disk; and hence, inside the complex number field. □

2 Discussion

The use of Rouché's theorem here makes the proof very short, but consequently makes this proof dependent on facts about complex numbers that are out of reach at the moment. However, it is possible to (with gratuitous hand-waving) motivate what this theorem is doing. In essence, we have compared $p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ to the simpler to understand polynomial $a_n x^n$. The polynomial $a_n x^n$ has a root $x = 0$ with multiplicity n; and what Rouché's theorem is actually providing is a rigorous way to say that $p(x)$ does not "differ" much from $a_n x^n$ in the complex number field. For example: any polynomial of odd degree must cross the x-axis once: since any odd degree polynomial reaches diagonally opposite quadrants, we can apply the Intermediate Value Theorem to see this fact. Similarly, if we start with $x^2 + 0x + 0 = 0$ and begin changing the coefficients of this quadratic, we can precisely track where the roots of this polynomial are going by use of the quadratic formula. In general, any polynomial of degree n can be compared to the polynomial $a_n x^n$ through examining the positions of their roots.

3 Examples

The Fundamental Theorem of Algebra is only applicable for non-zero polynomials of finite degree n. These hypotheses are essential:

- Zero polynomials:
  $p(x) = 0$ is clearly zero for all real numbers. Thus it is necessary for the polynomials to be nonzero.
- Non-polynomials: Right away, notice that the concept of "degree", while easily generalized, does not extend similarly for the Fundamental Theorem of Algebra:
  $f(x) = \sqrt{x}$ has one zero at $x = 0$, yet the "degree" of this function is $\frac{1}{2}$.
  $f(x) = \frac{1}{x}$ has no zeroes, with "degree" -1.
- Finite degree: a polynomial may have infinite degree, if we conceive of a polynomial that never ends:
  $p(x) = 1 + x + x^2 + ...$, where the ... denotes the pattern continuing forever. This polynomial has no roots, actually- complex or purely real.

The conclusion of Fundamental Theorem of Algebra must be complex roots. There are polynomials whose roots are of any type of number:

- Whole number: $p(x) = x - 1$ has root $x = 1$.
- Integer: $p(x) = x^2 - 1$ has roots $x = 1, -1$.
- Rational: $p(x) = 2x - 1$ has roots $x = \frac{1}{2}$.
- Irrational: $p(x) = x^2 - 2$ has roots $x = \sqrt{2}$.
- Pure imaginary: $p(x) = x^2 + 1$ has roots $x = i, -i$.
- Complex: $p(x) = x^2 - 2x + 2$ has roots $x = i + 1, i - 1$.

The Fundamental Theorem of Algebra does not give us any information *where* roots may be, only that they exist:

- $p(x) = x^4 - 3x^3 + 2x^2 + 2x - 4$ must have 4 complex roots, by the Fundamental Theorem of Algebra. In fact, those roots are $-1, 2, 1+i, 1-i$, but finding those roots is a highly nontrivial task.

The Fundamental Theorem of Algebra is actually valid for polynomials with complex coefficients, although the task of finding roots becomes even more difficult:

- $p(x) = 2ix^3 - (4 + i)x^2 + (2 - i)x + 2$ has roots $x = -2i, -\frac{1}{2}, 1$.