1 Homework Discussion: Wave Equation Stability

The crux of the proofs for FDM stability for the wave equation hinge on understanding how to take the absolute value of a complex number: $|z|^2 = z\bar{z}$. With this, recognize that when the term $\alpha + C\cos\varphi + iC\sin\varphi$ appears, it is describing in the complex plane a circle of radius C centered at $\alpha + 0i$ or $(\alpha, 0)$. You can carry this geometric thread and for any particular α determine the condition for C that will allow this circle to fit inside the unit circle on the complex plane, or you can work it using the above-mentioned absolute value definition:

$$\begin{aligned}
|\alpha + C\cos\varphi + iC\sin\varphi| &= ((\alpha + C\cos\varphi) + (iC\sin\varphi))\,((\alpha + C\cos\varphi) - (iC\sin\varphi)) \\
&= (\alpha + C\cos\varphi)^2 + (C\sin\varphi)^2 \\
&= \alpha^2 + 2\alpha C\cos\varphi + C^2\left(\cos^2\varphi + \sin^2\varphi\right) \\
&\leq \alpha^2 + 2\alpha C + C^2 \\
&= (\alpha + C)^2
\end{aligned}$$

Stability occurs when $(\alpha + C)^2 \leq 1$: once again, the condition here is that $C \leq 1 - \alpha$.

2 Elliptical PDE

2.1 Background

The prototypical elliptical PDE is the *Poisson equation*, used in various areas of physics and engineering. The problem is, given f, solve the following equation for u:

$$\nabla^2 u = f \tag{1}$$

Here, $\nabla^2 u$ represents the Laplacian of u; it is the gradient of the divergence of u, or $\nabla \cdot \nabla u$. The three dimensional example is $\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$. Keeping with our current investigation of FDM processes, we will analyze the two dimensional case; note that the derivatives here are spatial derivatives for (x, y).

The FDM setup will thus require us to partition a two dimensional spatial domain; so we will have parameters for uniform partition on x (Δx, or h), and a uniform (but not necessarily the same size) partition on y (Δy, or k). Given a problem domain, we essentially place a uniform net of rectangles over it; label the points at the corners of these rectangles by $u_{j,l}$.

With this setup our FDM operator $C_{j,l}$ will be defined as:

$$C_{j,l}u_{j,l} = \frac{1}{h^2}\left(u_{j+1,l} - 2u_{j,l} + u_{j-1,l}\right) + \frac{1}{k^2}\left(u_{j,l+1} - 2u_{j,l} + u_{j,l-1}\right)$$

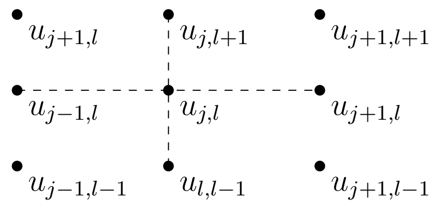

Figure 1: The 2-D FDM nodes

2.2 Convergence of FDM for elliptical PDE

Suppose the problem $\nabla^2 u = f$ is well-posed. It turns out that the elliptical PDE FDM will always converge.

Theorem 2.1. *The FDM for well-posed elliptical PDEs always converges.*

Note: This proof was left incomplete by the end of class, and will most likely be reviewed at the start of next class.

Proof. First, some labels: let x_j and y_l represent points in our partition; for now let us assume the problem domain is the unit square $[0,1] \times [0,1]$ so that x_j and y_l represent the coordinates of the uniform partitions in x and y respectively. Let us define the interpolated function $u(x,y)$ such that $u(x_j, y_l) = u_{j,l}$, where each $u_{j,l}$ is a value associated with a point in the 2-D partition. Finally, let $\bar{u} = \langle u_{j,l} \rangle$, the vector consisting of all the values, and let C represent the results vector $C\bar{u} = \langle C_{j,l} u_{j,l} \rangle$. With this interpretation of u, our theorem is re-interpreted as:

Theorem 2.2. *As $(\Delta x, \Delta y) \to (0,0)$, $\bar{u} \to u$ in L^∞ space.*

Notation: P is the partition of the problem domain, P^o represents the interior partition elements of the problem domain, and ∂P represents the boundary of the partition.

We first need a result on where the L^∞ maximum of $\bar{u}$ will be:

Theorem 2.3. $-C_{j,l} u_{j,l} \leq 0 \ \forall j,l \implies \max \|\bar{u}\|_\infty$ *is in* ∂P

Proof. Start with the definition of $C_{j,l} u_{j,l}$ and rearrange terms:

$$-C_{j,l} u_{j,l} \leq 0$$

$$-\left(\frac{1}{h^2} \left(u_{j+1,l} - 2u_{j,l} + u_{j-1,l} \right) + \frac{1}{k^2} \left(u_{j,l+1} - 2u_{j,l} + u_{j,l-1} \right) \right) \leq 0$$

$$\left(\frac{1}{h^2} + \frac{1}{k^2} \right) u_{j,l} \leq \frac{1}{2} \left(\frac{1}{h^2} (u_{j+1,l} + u_{j-1,l}) \right) + \frac{1}{2} \left(\frac{1}{k^2} (u_{j,l+1} + u_{j,l-1}) \right) \tag{2}$$

Now, suppose $u_{j,l}$ achieves its maximum on the interior (hence all of its neighbors will exist and have values). Because $u_{j,l}$ is the maximum here, we can push the inequality more: for example, we can use $u_{j,l} \geq u_{j+1,l},\ u_{j,l} \geq u_{j-1,l},$ and $u_{j,l} \geq u_{j,l-1}$:

$$\frac{1}{2}\left(\frac{1}{h^2}(u_{j+1,l}+u_{j-1,l})\right)+\frac{1}{2}\left(\frac{1}{k^2}(u_{j,l+1}+u_{j,l-1})\right)$$
$$\leq$$
$$\frac{1}{2}\left(\frac{1}{h^2}(u_{j,l}+u_{j,l})\right)+\frac{1}{2}\left(\frac{1}{k^2}(u_{j,l+1}+u_{j,l})\right)$$

In conjunction with (2) we have then that:

$$\left(\frac{1}{h^2}+\frac{1}{k^2}\right)u_{j,l} \leq \frac{1}{2}\left(\frac{1}{h^2}(u_{j,l}+u_{j,l})\right)+\frac{1}{2}\left(\frac{1}{k^2}(u_{j,l+1}+u_{j,l})\right) \tag{3}$$

$$\cancel{\frac{1}{h^2}u_{j,l}}+\frac{\cancel{1}}{\cancel{k^2}}u_{j,l} \leq \cancel{\frac{1}{h^2}u_{j,l}}+\frac{\cancel{1}}{\cancel{k^2}}\left(\frac{u_{j,l+1}+u_{j,l}}{2}\right)$$
$$u_{j,l} \leq \left(\frac{u_{j,l+1}+u_{j,l}}{2}\right)$$

Notice, however, that the expression on the right is the simple average of $u_{j,l+1}$ and $u_{j,l}$. Since we assumed that $u_{j,l}$ is the maximum, this is a contradiction: thus, so long as $u_{j,l}$ has all four of its neighbors (i.e., $u_{j,l} \in P^o$) it can not be the maximum. $u_{j,l}$ must achieve its maximum on the boundary. $\square$

Next we need the following result relating $\bar{u}$ and $C\bar{u}$:

Lemma 2.4. *If the boundary entries of $\bar{u}$ are all 0 then* $\|\bar{u}\|_\infty \leq \frac{1}{8}\|C\bar{u}\|_{\infty,o}$

Proof. We will make use of the following construct: we will define $v_{j,l}$, the parts of our partition that correspond to a function that is zero at the middle of our problem domain $[0,1]\times[0,1]$ and satisfies $\nabla^2 v = 1$ on the whole problem domain:

$$v_{j,l} = \frac{1}{4}\left((x_j-\frac{1}{2})^2+(y_l-\frac{1}{2})^2\right) \tag{4}$$

Define $\bar{v} = \langle v_{j,l}\rangle$ as the vector of these values, similar to before. It is possible to check by direct substitution that $C_{j,l}v_{j,l} = 1$ for any j,l. Remembering that $u_{j,l}$ may take on negative values, we start the next chain of deductions with the following equation:

$$0 \leq C_{j,l}u_{j,l} + \|C\bar{u}\|_{\infty,o}$$

Here the ∞, o mean "in the L^∞ norm (also known as the maximum value of $|C\bar{u}|$ for all j,l), in the interior of P".

If this is the case, then we can use the fact that $C_{j,l}v_{j,l} = 1$ to get the following inequality:

$$0 \le C_{j,l}u_{j,l} + \|C\bar{u}\|_{\infty,o} C_{j,l}v_{j,l} \tag{5}$$

Since the operation $C_{j,l}$ is linear, we can rewrite (5) as:

$$0 \le C_{j,l}\left(u_{j,l} + \|C\bar{u}\|_{\infty,o} v_{j,l}\right)$$

Using (2.3) we have that $\max \|(\bar{u} + \|C\bar{u}\|_{\infty,o}\bar{v})\|_{\infty}$ is on the boundary. Since $v_{i,j}$ is always positive, this is the same as saying $\max \|(\bar{u} + \|C\bar{u}\|_{\infty,o})\|_{\infty}$ is on the boundary.

By a similar argument, we can demonstrate that $\min \|(\bar{u} - \|C\bar{u}\|_{\infty,o})\|_{\infty}$ is on the boundary. □

That was just the proof of the lemma above, we still need to prove that the FDM converges. □

3 Finite Element Method

3.1 Derivation of the FEM Matrix

Given the problem $\nabla^2 f = 0$ we seek to find the *weak solution*; that is, φ such that $\int \nabla^2 \varphi\psi = 0$ for some ψ. In this situation, we will be searching for an FEM solution which takes the form of a piecewise polynomial u_h associated with some partitioning of the problem domain and basis polynomials on the partition. The basis polynomials are the "pieces"; each is 0 for all points in the partition except for at the point it is dual to.

Now, if $u_h = \sum \alpha_p \varphi_p$ then our weak solution takes the form

$$\int_D \left(\nabla^2 \sum \alpha_p \varphi_p\right) \psi_b = 0$$

Note the linearity of the integral and the derivatives means we can switch the sum:

$$0 = \sum \alpha_p \left(\int_D \nabla^2 \varphi_p \psi_b \ \forall b\right)$$

This is a linear system; thus we can represent it as $K\vec{\alpha} = 0$ where every index represents an element of the partition. So we need to learn what the entries of K will be: what, exactly, are the $\int_D \nabla^2 \varphi_p \psi_q$?

First of all, we are really summing over elements. On any particular element, there are 4 basis polynomials on each. Taking advantage of the dual property, we have that:

$$\sum_p \alpha_p \int_D \nabla^2 \varphi_p \psi_b \ \forall b = \sum_e \int_{E_e} \nabla^2 \varphi_p^e \psi_q^e$$

Here the p and q indices stand for a number 1 to 4, the four basis polynomials associated with each element (one for each corner). Practically this means we can replace these entries with N_i^e and N_j^e, where i and j are indices from 1 to 4. Now the following deduction is applied: since $\int_{E_e} \nabla^2 N_i^e N_j^e = \int_{E_e} \nabla \cdot \nabla N_i^e N_j^e$, we can apply first integration by parts and then Stokes' Theorem:

$$\begin{aligned}
0 = \sum_e \int_{E_e} \nabla^2 N_i^e N_j^e &= \sum_e \int_{E_e} \nabla(\nabla N_i^e) N_j^e \\
&= \sum_e \left(\int_{E_e} \nabla(\nabla\varphi)\psi - \int_{E_e} \nabla\varphi\nabla\psi \right) \\
&= \sum_e \int_{\partial E_e} \frac{\partial\varphi}{\partial n}\psi - \sum_e \int_{E_e} \nabla\varphi\nabla\psi = 0
\end{aligned}$$

In other words, $\sum_e \int_{\partial E_e} \frac{\partial\varphi}{\partial n}\psi = \sum_e \int_{E_e} \nabla\varphi\nabla\psi$. This is how we will implement our boundary conditions as well: for elements on the boundary of the problem domain, the $\frac{\partial\varphi}{\partial n}$ term will correspond to our fluid flows through the boundary of the problem domain.

3.2 Homework or: How I Learned to Stop Worrying and Love the FEM

Given the channel and nodes data, the first step will be to load the elements in one by one into your program. The very first task the program should do is to read each element in the file and draw it, thus testing the correct implementation of element record reading and element-node lookup functions. Once the underlying elements have been drawn, the next step is to begin coding the basis polynomials. Each basis polynomial should take as input the four coordinates corresponding to the bottom left and upper right coordinates of the element: (α, β) and (γ, δ) should be variables in the basis polynomial used to construct its dual relationship. On top of that, each basis polynomial accepts as variables (x, y). So a given basis polynomial is one such as $N_1(\alpha, \beta, \gamma, \delta, x, y) = \frac{(\gamma - x)}{(\gamma - \alpha)} \frac{(\delta - y)}{(\delta - \beta)}$.

After that, the next step is to initialize a square matrix whose size is given by the number of nodes: this will carry our approximation for u_h for every node in the partition. Call it K; the task will be to loop over all elements in the partition scheme and load into K the appropriate integrals given in the derivation: each integral $\int_{E_e} \nabla\varphi\nabla\psi$ will go into K, where φ and ψ are each one of the four basis polynomials (so there will be sixteen entries total per element).

The remaining steps have yet to be expanded on.