1 Multi-Index Notation

If we have n distinct variables $x_1, x_2, ...$ and with them n distinct exponents $\alpha_1, \alpha_2, ...$, we can shorthand these polynomials:

Let $X = (x_1, x_2, x_3 ...)$ represent all the variables; let $\alpha = (\alpha_1, \alpha_2, ...)$ represent the exponents in a list. The *length n multi-index* is

$$X^\alpha := \prod x_i^{\alpha_i}$$

Here $\forall i \alpha_i \geq 0$. Thus the generic n dimensional polynomial can be given as

$$\sum_{|\alpha| \leq m} a_\alpha X^\alpha$$

This is presented primarily as a shorthand for personal use to simplify writing these complicated polynomials.

2 Differentiation as a Bounded Operator

Consider the space of all polynomials over $[0, 1]$: $\mathcal{P}[0, 1]$ as a normed linear space with norm $\|p\|_\infty = \sup_{x \in [0,1]} |p(x)|$. Define D as the derivative; we have already seen that D is a linear operator on $\mathcal{P}[0, 1]$. Is D bounded?

D is not a bounded operator: consider $\|Dx^n\|_\infty = n\|x^{n-1}\|$. There is no value $M < \infty$ such that $\|Dx^n\|_\infty \leq M\|x^{n-1}\|$ (since, if $x = 1$, then $\|Dx^n\|_\infty = n$ which has no bound.

3 Polynomial Interpolation Again

Instead of Vandermonde matrices, consider interpolation using the Lagrange basis; in other words, functions $l_i(x)$ such that $l_i(x_j) = 1$ if $j = i$ and $l_i(x_j) = 0$ if $j \neq i$. With such a basis the polynomial can be expressed as

$$\sum_{k=0}^{n} \alpha_i l_i(x)$$

Assuming some partition $x_0, x_1, ...$, we will define the Newton Form of this expression as being of the form

$$\sum_{k=0}^{n} c_k \prod_{j=0}^{k} (x - x_j)$$

To see why this can be a viable form of the polynomial, the first few products $\prod(x - x_j)$ are given:

- $\prod_0 = 1$

- $\prod_1 = x - x_0$
- $\prod_2 = (x - x_0)(x - x_1)$

Note that this implies $\prod_i(x) = 0$ if $x = x_0, x_1, ..., x_{i-1}$. The c_k are various constant factors which we will derive next.

In order for the Newton Form to agree with the polynomial we have from the Lagrange basis approach, we will invent new notation for c_k. From the above thought on $\prod_i$, note that c_k will really depend on the values $x_0, x_1, x_2, ..., x_k$ and f. In that vein we define *divided-differences* notation as

$$c_k = [x_0, ..., x_k]f$$

To interpret this: $[x_0]f = f(x_0)$ first. Now, $f(x_1) = c_0 \prod_0 + c_1 \prod_1 = c_0 + c_1(x - x_0)|_{x=x_1}$; with $c_0 = [x_0]f = f(x_0)$ we can rewrite this as $f(x_1) = f(x_0) + c_1(x_1 - x_0)$ which leads us to the conclusion $c_1 = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$

And so on... this quickly gets out of hand, but let's try to keep some facts straight: $[x_0, ..., x_k]f$ will be the coefficient of x^n in our final polynomial: this argument proceeds by both induction and also by arguing from degree terms- using the Lagrange basis we have one expression for x^n's coefficient which must therefore be the same as what we've called $[x_0, ..., x_k]f$. The full details of this argument are highly algebraic in nature and are fully exposed in the text. The next divided difference can be determined either by the technique above or by employing the *recursive* formula for divided differences:

- $[x_0, x_1] = \dfrac{[x_1] - [x_0]}{x_1 - x_0}$
- $[x_0, x_1, x_2] = \dfrac{[x_1, x_2] - [x_1, x_0]}{x_2 - x_0}$
- $[x_0, x_1, x_2, x_3] = \dfrac{[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_3 - x_0}$

Another way to think of these is as discrete approximations to the derivatives: $[x_0] = f(x_0)$ is the 0th derivative; $[x_0, x_1] = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$ is the 1st derivative approximated, and so on.

3.1 Multivariable Interpolation

Extending the divided-differences formula to multivariable polynomial interpolation: again, let $[x_0][y_0]f = f(x_0, y_0)$. From here we build up by following the mold already set: $[x_0, x_1][y_0]f = \dfrac{f(x_1, y_0) - f(x_0, y_0)}{x_1 - x_0}$, $[x_0][y_0, y_1]f = \dfrac{f(x_0, y_1) - f(x_0, y_0)}{y_1 - y_0}$, and so on.

4 Fourier Transformation

If $f \in L^1(\mathbb{R}) \bigcap L^2(\mathbb{R})$ then we can define the *Fourier transform* of f by the following:

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\ dx$$

The effect of the transform is to take f from its original domain to a new domain: the frequency domain. There is also defined a function $g(x) = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}\ dk$ which has the following effect: $f = g$ almost everywhere. This is the subject of the *Inversion theorem*, which establishes that the Fourier transform is a reversible transformation: f can be transformed into $\hat{f}$, which can then be transformed back into g.

A few other key theorems: Plancherel's theorem states that the Fourier transform is a linear map; additionally, $\|\hat{f}\|_2 = \|f\|$ so it is an isometry. This has the effect of saying that the space of transformed functions and the space of functions are identical; that is, we need not fear losing any meaningful information by applying the Fourier transform.

There's the convolution theorem: first, we define the *convolution* of f and g as $(g*f)(x) = \int_{-\infty}^{\infty} f(x)g(x-x')\ dx'$. The theorem states that $\hat{g}(k)\hat{f}(k) = g(x) \hat{*} f(x)$.

There's this identity: $\hat{f}'(k) = ik\hat{f}(k)$. This is used in a concrete instance in the solution to the heat equation.

4.1 Solving 1-D Heat Equation

The underlying differential equation is

$$f(t,x) = \frac{\partial f}{\partial t} = \alpha \frac{\partial^2 f}{\partial x^2}$$

Applying a Fourier transform with respect to x gives us that

$$\frac{\partial}{\partial t}\hat{f}(t,k) = -\alpha k^2 \hat{f}(t,k)$$

We can solve this differential equation in t in terms of $\hat{f}$: a function whose derivative is a multiple of itself is the exponential. Thus

$$\hat{f}(t,k) = e^{-\alpha k^2 t} f_0(k)$$

So from here we transform back: the solution (almost everywhere) is going to be

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\alpha k^2 t} f_0(k)e^{ikt} dk$$

With appropriate assumptions on initial conditions we have produced the family of solutions to the 1-D heat equation.