1 Crank-Nicolson Finite Difference Approximation

1.1 Review

The 1-D heat equation is given by the following differential equation on $u(x,t)$:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} \tag{1}$$

The way the differential equation works is that any analytical solution to this differential equation $u(x,t)$ will be uniquely determined by $u(x,0) = u_0$, the initial value. This is similar to initial value problems from integral calculus. One numerical technique for approximating the solution function $u(x,t)$ given some initial conditions u_0 is the finite difference method. As usual, let v_i^n represent the value of our approximation at the ith spatial point at the nth time step. That is, $\vec{v}^n = (v_0^n, v_1^n, v_2^n, ..., v_K^n)$ where the n denotes not exponentiation but the nth time step, and the v_i elements are all values associated to a specific spatial point in the spatial partition. I choose to use v_i^n here because I do not want to confuse what u_0 means: u_0 is the description of the function $u(x,0)$, while v_0^n is the description of a value over some point on the partition at time step n. These are very different objects; u_0 is a function, v_0^n is a quantity.

1.2 Derivation

First, we use a forward time and central space approximation to the derivatives:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = a\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} \tag{2}$$

This approximation comes about by replacing the objects in (1) with their discrete counterparts. The right hand term is derived by considering the discrete approximation of $\frac{\partial}{\partial x}u'(x,t)$. Next, consider the backwards time and central space approximations to the derivative:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = a\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2} \tag{3}$$

The change here is that the terms are all in the context of u_k^{n+1} instead of u_k^n. The goal with Crank-Nicolson is to blend these two approximations together: the *Crank-Nicolson method* in this context is the following finite difference approximation to (1):

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{1}{2}\left(a\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} + a\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2}\right) \tag{4}$$

Employ a bit of algebra to collect all u_k^{n+1} terms on the left side and all u_k^n terms on the right side (so that there will be a relationship between the current time step and the future time step). Start by letting $\lambda = \dfrac{a\Delta t}{\Delta x^2}$

$$\begin{aligned}
\frac{u_i^{n+1} - u_i^n}{\Delta t} &= \frac{1}{2}\left(a\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} + a\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2}\right) \\
u_i^{n+1} &= u_i^n + \frac{\lambda}{2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n + u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right) \\
u_i^{n+1} - \frac{\lambda}{2}\left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right) &= u_i^n + \frac{\lambda}{2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) \\
\frac{-\lambda}{2}u_{i+1}^{n+1} + (1+\lambda)u_i^{n+1} + \frac{-\lambda}{2}u_{i-1}^{n+1} &= \frac{\lambda}{2}u_{i+1}^n + (1-\lambda)u_i^n + \frac{\lambda}{2}u_{i-1}^n \qquad (5)
\end{aligned}$$

Thus if we conceive of our approximations as vectors, where $\vec{u}^n$ represents the state of our approximation at the nth time step, we have the following relationship:

$$\begin{bmatrix} 1+\lambda & \frac{-\lambda}{2} & 0 & \dots & 0 \\ \frac{-\lambda}{2} & 1+\lambda & \frac{-\lambda}{2} & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \frac{-\lambda}{2} & 1+\lambda & \frac{-\lambda}{2} \\ 0 & 0 & 0 & \frac{-\lambda}{2} & 1+\lambda \end{bmatrix} \begin{bmatrix} u_0^{n+1} \\ u_1^{n+1} \\ \vdots \\ \vdots \\ u_K^{n+1} \end{bmatrix} = \begin{bmatrix} 1-\lambda & \frac{\lambda}{2} & 0 & \dots & 0 \\ \frac{\lambda}{2} & 1-\lambda & \frac{\lambda}{2} & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \frac{\lambda}{2} & 1-\lambda & \frac{\lambda}{2} \\ 0 & 0 & 0 & \frac{\lambda}{2} & 1-\lambda \end{bmatrix} \begin{bmatrix} u_0^n \\ u_1^n \\ \vdots \\ \vdots \\ u_K^n \end{bmatrix} \qquad (6)$$

The expression (6) is a bit unwieldy, so we will recast it in terms of matrices A and B; and we will use vector notation for $\vec{u}^{n+1}$ and $\vec{u}^n$:

$$B\vec{u}^{n+1} = A\vec{u}^n \qquad (7)$$

Thus, to write this in terms of a state change matrix we have that:

$$\vec{u}^{n+1} = B^{-1}A\vec{u}^n \qquad (8)$$

This is the Crank-Nicolson method.