1 Exterior Products and Determinants

Let $\bigwedge V$ be the exterior product space as before with basis $\{v_i\}$. We showed already that for any two vectors we can rewrite the product: $\sum \alpha_{1j} v_j \wedge \sum \alpha_{2k} v_k = \sum_{\sigma \in S_n} (-1)^{\sigma} \prod_{i \in 1}^{n} \alpha_{i\sigma(i)} (v_1 \wedge \cdots \wedge v_n)$. Notice that if we indexed the coefficients and thought of them as matrix entries in $A = [\alpha_{ij}]$ that this is precisely the definition of the determinate of $\det A$. Let's explore the implications of this connection.

If we have a linear map $L : V \to V$ with $L(v) = u$ and matrix representation $A = [L]$. We say that there is a map $L^\wedge$ induced by L: $L^\wedge \bigwedge^n V \to \bigwedge^n V$ with $L^\wedge(v_1 \wedge \cdots \wedge v_n) = (Lv_1 \wedge Lv_2 \wedge \cdots \wedge Lv_n)$. By working with this definition, it is easily shown that $u_1 \wedge \cdots \wedge u_n = \det A (v_1 \wedge \cdots \wedge v_n)$. Thus we can identify $[L] = A$ and $[L^\wedge] = \det A$.

1.1 Determinant Rules

Let $B = [N]$ be another map and matrix representation.

Claim: $\det B \det A = \det BA$

Proof. Call $A = [NL]$; by our definition of determinant $[(NL)^\wedge] = \det BA$. On the one hand, $[NL] = [N][L]$. $[(N \circ L)^\wedge] \cong (NL)^\wedge (v_1 \wedge \cdots \wedge v_n) = (NLv_1 \wedge \cdots \wedge NLv_n)$. At this point we make use of the $N^\wedge$ definition: this equals $N^\wedge (Lv_1 \wedge \cdots \wedge Lv_n)$; similarly, we can pull out $L^\wedge$ to get $N^\wedge L^\wedge (v_1 \wedge \cdots \wedge v_n)$. Thus $(N \circ L)^\wedge = N^\wedge L^\wedge$; so $\det BA = [(NL)^\wedge] = [N^\wedge L^\wedge] = [N^\wedge][L^\wedge] = \det B \det A$.

As a corollary, note that $(N \circ L)^\wedge \in \operatorname{Hom} \bigwedge^n V$. □

Claim: $\det I = 1$

Proof. $I^\wedge (v_1 \wedge \cdots \wedge v_n) = v_1 \wedge \cdots \wedge v_n = 1 \cdot v_1 \wedge \cdots \wedge v_n$. Thus $\det I = [I^\wedge] = [1] = 1$ where we identify the 1 by 1 matrix with a constant. □

Claim: $\det A^{-1} = (\det A)^{-1}$

Proof. $1 = \det I = \det AA^{-1} = \det A \det A^{-1}$. Hence $\det A^{-1} = (\det A)^{-1}$ by division on constants. □

Alternatively, notice carefully what we have done by defining the determinant in this way: if we take these matricies to be elements of the general linear group $GL(\mathbb{R})$ then the relationship $[L] = A \implies [L^\wedge] = \det A$ is in fact a group homomorphism. Now the properties become clear: identity and cancellation are as given by group properties.

Claim: $\det A \neq 0 \iff A$ nonsingular.

Proof. We have already shown the other case by the result on $\det A^{-1}$. Now, the $\implies$ case: if L is a singular map then this is equivalent to the basis elements $\{v_i\}$ mapping under L to a dependent set $\{u_i\}$. If so, then $L^\wedge (v_1, ..., v_n) = 0$ (see our work earlier when defining the exterior product; the exterior product of a dependent set is zero). Hence $[L^\wedge] = A = [0]$. □

1.2 Exterior Product & Elementary Row Operations

We can reinterpret the elementary matrix row operations now. Type I: swapping rows (identify with the transposition (ij):

$$\begin{aligned} L^{\wedge}\left(v_1, ..., v_n\right) &= \left(Lv_1, ..., Lv_n\right) \\ &= v_{\tau(1)} \wedge \cdots \wedge v_{\tau(n)} \\ &= -\left(v_1 \wedge \cdots \wedge v_n\right) \end{aligned}$$

Hence $[L^{\wedge}] = -1$; thus the type I operation changes the sign of the determinant.

Type II: multiplication of a row by c constant:

$$\begin{aligned} L^{\wedge}\left(v_1, ..., v_n\right) &= \left(v_1 \wedge \cdots \wedge cv_i \wedge \cdots \wedge v_n\right) \\ &= c\left(v_1 \wedge \cdots \wedge v_n\right) \end{aligned}$$

Thus $[L^{\wedge}] = c$; so the type II operation changes the determinant by multiplication by c.

Type III: linear combination of rows $(ci + j)$

$$\begin{aligned} & L^{\wedge}\left(v_1 \wedge \cdots \wedge cv_i + v_j \text{ in the } j\text{th position } \wedge \cdots v_n\right) \\ & = \left(v_1 \wedge \cdots \wedge v_i \text{ in the } j\text{th position } \wedge \cdots \wedge v_n\right) + \left(v_1 \wedge \cdots \wedge v_n\right) \\ & = 0 + \left(v_1 \wedge \cdots \wedge v_n\right) \end{aligned}$$

Hence $[L^{\wedge}] = 1$. Hence the linear combination of rows does not change the determinants.