1 Exterior Product

1.1 Definition

Let V be a vector space; $\bigotimes^m V = V \otimes V \otimes \cdots \otimes V$ be the tensor product of m many copies. At this point we will construct a new space by taking out the antisymmetric tensors. Call $\mathcal{W} = \text{Span}\left\{\text{all tensors of the form } v_1 \otimes \cdots \otimes v_m - v_{\tau(1)} \otimes \cdots \otimes v_{\tau(m)}\right\}$ where $\tau(i)$ is an arbitrary transposition of two elements (remember that we can build up permutation by composition of transpositions). We will denote $\bigwedge^m V$ by $\bigotimes^m V/\mathcal{W}$; elements take the form of equivalence classes $v_1 \otimes v_2 \otimes \cdots \otimes v_n + \mathcal{W} = v_1 \wedge v_2 \wedge \cdots \wedge v_n$. Sometimes we denote this space as the *Grassman Space*; the algebra on these elements is called the *Grassman Algebra* or the *Exterior Algebra*. One of its uses can be found in permitting differential geometry to work- the algebra of differential forms is the exterior algebra. In a concrete geometric sense: if we have two dimensional vectors, there is the two dimensional vector space they create (assuming they are not collinear). Additionally, for any two vectors in this two dimensional vector space, there can be defined the parallelogram (one vector multiplied along another) which will have properties such as area, orientation, facing. The exterior product of the two dimensional vector space is hence the space of all possible parallelograms.

1.2 Basic Properties

Claim: If $v_i = v_j$ then $v_1 \wedge v_2 \wedge \cdots \wedge v_m = 0$

Proof. If we use the transposition $\tau = (ij)$ observe that this has signature of -1 (single transpositions have signature -1; arbitrary permutations have signature $(-1)^n$ where n is the number of required transpositions). It is not too hard to check, by nature of the construction, that for any permutation σ that the permutation $v_{\sigma(1)} \wedge v_{\sigma(2)} \wedge \cdots \wedge v_{\sigma(m)} = \text{sgn}(\sigma)(v_1 \wedge v_2 \wedge \cdots v_n)$; this naturally implies that if the element in the ith position and in the jth position then we have that $v_1 \wedge v_2 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_j \wedge \cdots \wedge v_n = v_{\tau(1)} \wedge v_{\tau(2)} \wedge \cdots \wedge v_{\tau(i)} \wedge \cdots \wedge v_{\tau(j)} \wedge \cdots \wedge v_{tau(n)} = -v_1 \wedge v_2 \wedge \cdots \wedge v_j \wedge \cdots \wedge v_i \wedge \cdots \wedge v_n = -v_1 \wedge v_2 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_j \wedge \cdots \wedge v_n$. The first object is equal to itself times negative 1: this can only happen if the object as a whole is equal to a 0-object (possibly a zero vector of some length). It will turn out to be a dimension 1 0 vector- what we will associate with just the number 0. □

Claim: If $\{v_1, ..., v_n\}$ are a dependent set, then $v_1 \wedge v_2 \wedge \cdots \wedge v_n = 0$.

Proof. By dependence we can assume without loss of generality that $v_n = \sum_{i=1}^{n-1} \alpha_i v_i$. Now: rewrite the wedge product as $v_1 \wedge \cdots \wedge v_{n-1} \wedge \sum_{i=1}^{n-1} \alpha_i v_i$; if we go back to the definition of this wedge product we see that it exhibits multilinearity owing to its tensor product construction.

Hence the wedge product $v_1 \wedge v_2 \wedge \cdots \wedge v_n = \sum_{i=1}^{n-1} \alpha_i (v_1 \wedge v_2 \wedge \cdots \wedge v_{n-1} \wedge v_i)$. Observe that for any i the wedge product in that sum must equal zero: since for any i that wedge product contains two copies of that element. Hence this whole sum is zero; $v_1 \wedge v_2 \wedge \cdots \wedge v_n = 0$. □

Claim: If $\dim V = m$ and $n > m$, then $\bigwedge^n V = 0$

Proof. If you take the wedge products of n spaces but they each only have dimension m with $n > m$ then it is impossible to choose an independent set $\{v_i, ..., v_n\}$. Thus the wedge product must equal zero. □

1.3 Dimension of General Exterior Product

We can actually make a general statement on the dimension of the exterior product:

Claim: If $\dim V = m$ then $\dim \bigwedge^n V = \binom{m}{n}$.

Proof. Let $\{v_i, ..., v_n\}$ be a basis for V; thus we'll consider the wedge product of arbitrary vectors $u_1 \wedge u_2 \wedge \cdots \wedge u_n$ where $u_i = \sum \alpha_{ij} v_j$ and attempt to construct a basis for $\bigwedge^n V$. By doing show we can count the number of basis elements, and determine the dimension. The wedge product of $\{u_i\}$ is $u_1 \wedge u_2 \wedge \cdots \wedge u_n = \sum \alpha_{1j} v_j \wedge \sum \alpha_{2j} v_j \wedge \cdots \wedge \sum \alpha_{nj} v_j$. We want to be able to operate multilinearity, but in order to reduce this down into a sum with a $v_1 \wedge v_2 \wedge \cdots \wedge v_n$ term in it we'll need to do some analysis on the arbitrary nature of the ordering of these terms. Let $\Gamma(n)$ denote all possible sequences of length n of m letters.

With this set defined, notice that we can formulate a basis for $\bigotimes^n V$:

$$\left\{ v_{\gamma(1)} \otimes \cdots \otimes v_{\gamma(m)} = v^{\gamma} | \forall \gamma \in \Gamma(n) \right\} \text{ is a basis for } \bigotimes^n V$$

Remember that $\dim \bigotimes^n V = m^n$ for $\dim V = m$. Now: if we return to our product $\sum \alpha_{1j} v_j \wedge \sum \alpha_{2j} v_j \wedge \cdots \wedge \sum \alpha_{nj} v_j$ we will use multilinearity to try to collect all of the terms. First, we can gather all of the α terms together: to see how this is done, consider the simple case of $n = 2$:

$n = 2$ case:

$\sum_{i=1}^{m} \alpha_{1i} v_i \wedge \sum_{j=1}^{m} \alpha_{2j} v_j$

Immediately we can see cancellation will happen between some of these terms in this product. For example, whenever $i = j$ we have a previous result that says their wedge product will equal zero (since $v_i \wedge v_i = 0$). Moreover, there will be cancellation between matching pairs of $v_i \wedge v_j$ and $v_j \wedge v_i$.

For the $i = 1$ term: $\alpha_{11} v_1 \wedge \sum_{j=1}^{m} \alpha_{2j} v_j = \sum_{j=1}^{m} \alpha_{11}\alpha_{2j}(v_1 \wedge v_j)$. This follows from the multilinearity of our construction. For $i = 2$ we will have, similarly, $\sum_{j=1}^{m} \alpha_{12}\alpha_{2j}(v_2 \wedge v_j)$. And so on, for we will have m of these terms and they will all be summed together. Hence we will have a double sum:

$\sum_{i=1}^{m}\sum_{j=1}^{m} \alpha_{1i}\alpha_{2j}(v_i \wedge v_j)$. When $i = j$ the term in the sum is zero because of $v_i \wedge v_i = 0$; the other terms will collapse together (since $v_i \wedge v_j = -v_j \wedge v_i$), and we are left with the sum $\sum_{\substack{i,j \\ i<j}} \alpha^{\wedge}_{1i}\alpha^{\wedge}_{2j}(v_i \wedge v_j)$ where the α terms have changed owing to the collection of $i > j$ terms. Let's push this back up to the general case now that we see what's happening: the wedge product of sums turns into a multiple sum with the α terms entering a product.

So if we have a general n length wedge product, we can still factor out the first entry: $\sum_{i=1}^{m} \alpha_{1i} v_i \wedge \cdots \wedge \sum_{i=1}^{m} \alpha_{ni} v_i = \sum_{j=1}^{m} \alpha_{1j} \left(v_1 \wedge \cdots \wedge \sum_{i=1}^{m} \alpha_{ni} v_i \right)$ (it's a bit confusing, but the is are all indexing separate things. Now we continue factoring out alphas from the wedge product, just like as in the two product case: $= \sum_{\substack{j,k \\ j<k}} \alpha^{\wedge}_{1j}\alpha^{\wedge}_{2k} \left(v_j \wedge v_k \wedge \cdots \wedge \sum \alpha_{ni} v_i \right)$. Be aware that since each element of this wedge product is going over all elements of the basis of V that we do not have any explicit ordering of these factored-out wedge products; in other words, v_j and v_k have no relationship between them yet. If we continue with this product, factoring out terms, we see that:

- All the scalar terms are going to end up outside the wedge product
- The sum becomes a multiple sum of n seperate indices
- The wedge product inside the multiple sum is arbitrarily arranged

If we piece these facts together we conclude that we can express the whole thing in terms of a sum over $\gamma \in \Gamma(n)$ such that the sequence is increasing (remember the two product case where we showed that the multiple sum ends up being $\sum_{\substack{j,k \\ j<k}}$; this carries over into multiple indices); denote the set of increasing sequences of length n over m letters as $\Lambda(n)$:

$$\sum_{\gamma \in \Lambda(n)} \overline{\alpha_j} \left(v_{\gamma(1)} \wedge \cdots \wedge v_{\gamma(n)} \right)$$

Thus for n arbitrary vectors $\{u_i\}$ we can express them as the sum $\sum_{\gamma \in \Lambda(n)} \overline{\alpha_j} \left(v_{\gamma(1)} \wedge \cdots \wedge v_{\gamma(n)} \right)$;

the conclusion is that

$$\sum_{\gamma \in \Lambda(n)} \left(v_{\gamma(1)} \wedge \cdots \wedge v_{\gamma(n)} \right) \text{ must span } \bigwedge^{n} V$$

Finally, since we have a spanning set for the space $\bigwedge^{n} V$, we can ask: what dimension is $\bigwedge^{n} V$? Looking at what we have above, we conclude that $\dim \bigwedge^{n} V$ is the same as the size of $\Lambda(n)$. If you recall your combinatorics, this is precisely $\binom{m}{n}$ □

1.4 Universal Property

Claim: For an exterior product generated on n vector spaces $\bigwedge^n V_i \neq 0$, the resultant space satisfies the universal property.

Proof. We must show that for any n-linear map $\varphi(v)$ (that is, a mapping in n separate variables) that there exists linear maps L and a unique $L^\wedge$ such that $L^\wedge(L(v)) = \varphi(v)$. To do so, we will use the following diagram as a framework: We will make use of the

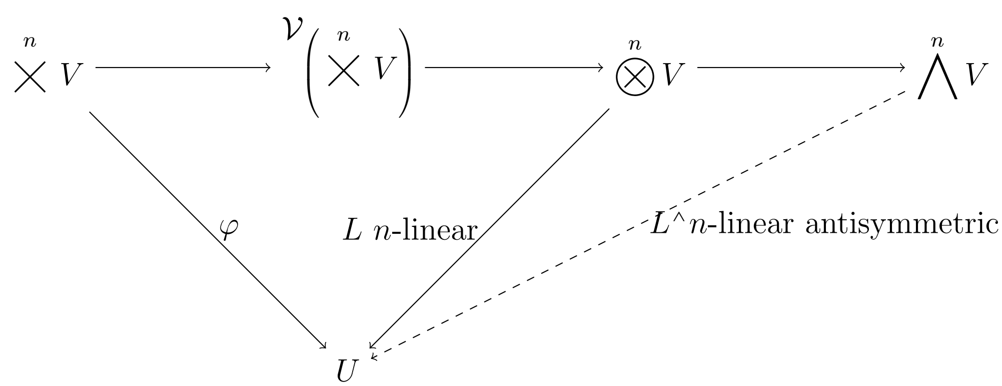

Figure 1: Relations between spaces and the exterior product

universality of the tensor product. Consider the following map: $S : \bigtimes^{n} V \to \bigotimes^{n} V$ defined by $S(v_1, v_2, ..., v_k) = \sum_{\sigma \in S_n} (-1)^\sigma P(\sigma)(v_1 \otimes \cdots \otimes v_n)$ where $P(\sigma)(v_1 \otimes \cdots \otimes v_n) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$ is a permutation of tensors. We wish to show that this fulfills the requirements we want for an antisymmetric n-linear map. For an arbitrary transposition of inputs, τ: evaluate

$S(v_{\tau(1)}, ..., v_{\tau(n)})$:

$$\begin{aligned}
S(v_{\tau(1)}, ..., v_{\tau(n)}) &= \sum_{\sigma \in S_n} (-1)^\sigma P(\sigma) \left(v_{\tau(1)} \otimes \cdots \otimes v_{\tau(n)}\right) \\
&= \sum_{\sigma \in S_n} (-1)^\sigma P(\sigma\tau) \left(v_1 \otimes \cdots \otimes v_n\right) \\
&= \sum_{\sigma \in S_n} (-1)^{\sigma\tau^{-1}} P\left(\sigma\tau^{-1}\right) \tau \left(v_1 \otimes \cdots \otimes v_n\right) \\
&= \sum_{\sigma \in S_n} (-1)^\sigma (-1) P(\sigma) \left(v_1 \otimes \cdots \otimes v_n\right) \\
&= -\sum_{\sigma \in S_n} (-1)^\sigma P(\sigma) \left(v_1 \otimes \cdots \otimes v_n\right)
\end{aligned}$$

Hence the function S is antisymmetric, and is the linear map we are looking for. □