1 Tensor Products and Hom

Notation: we use π where the book uses $\odot$

Let U, V vector spaces of dimension n; basis sets $B = u_i; i = 1, ..., n$ and $C = v_j, j = 1, ..., n$. Consider the relationship between $\text{Hom}(U \otimes V)$ and $\text{Hom}(U) \otimes \text{Hom}(V)$: we will establish an isomorphism between the two.

Let $L \in \text{Hom}(U)$, $N \in \text{Hom}(V)$. Set $L\pi N(u \otimes v) = L(u) \otimes N(v)$; $L\pi N$ is an element of $\text{Hom}(U \otimes V)$ and $L(\cdot) \otimes N(\cdot)$ is an element of $\text{Hom}(U) \otimes \text{Hom}(V)$.

First concern: Not every element of $U \otimes V$ may be of the form $u \otimes v$; that is, there may be some *non-decomposable* elements of the form $u_1 \otimes v_1 + u_2 \otimes v_2$ that do not simplify into $u \otimes v$. To dance around this neatly, we instead patch up our definition of $L\pi N$: we will define it over the *basis elements* of $U \otimes V$, $u_i \otimes v_j$ for i, j; since now $L\pi N$ is a linear transformation on the basis of a space it is now the linear transformation of all of that space $U \otimes V$; thus we have that $L\pi N$ is a unique element of $\text{Hom}(U \otimes V)$ (as determined by what $L\pi N$ does to the basis elements). This is going back to the theory of linear transformations on vector space basis.

For an isomorphism we need a bijective linear map between two spaces. Can we produce a linear map $\mathcal{L} : \text{Hom}(U) \otimes \text{Hom}(V) \to \text{Hom}(U \otimes V)$? Using the universal property, we can:

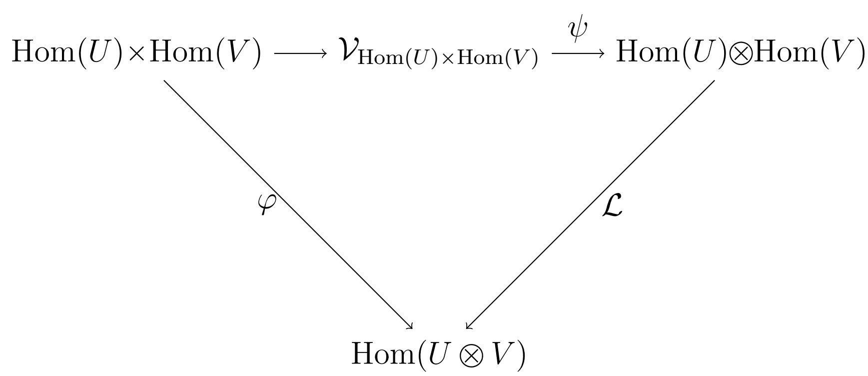

Figure 1: Existence of a commuting map $\mathcal{L}$

If we can find a bilinear map φ then the universal property of the tensor product gives us the existence of the linear map $\mathcal{L}$. As it turns out, the map $\varphi(L, N) = L\pi N(u_i \otimes v_j)$ is exactly that bilinear map. It's easy enough to check linearity:

$$\begin{aligned}\varphi(L_1 + L_2, N)(u_i \otimes v_j) &= (L_1 + L_2)u_i \otimes Nv_j \\ &= L_1 u_i \otimes Nv_j + L_2 u_i \otimes Nv_j \\ &= \varphi(L_1, N) + \varphi(L_2, N)\end{aligned}$$

Similar arguments hold for the remaining bilinear conditions; they all hinge upon the linearity of the tensor product on elements of U and V (remember that $Lu \in U$ and $Nv \in V$ since $L \in \text{Hom}(U)$ and $N \in \text{Hom}(V)$).

Thus the bilinear map φ gives us a unique linear map $\mathcal{L} : \mathrm{Hom}(U) \otimes \mathrm{Hom}(V) \rightarrow \mathrm{Hom}(U \otimes V)$.

Now we have to determine that $\mathcal{L}$ is a bijective map; this will turn out to be very nontrivial. The dimension of the domain and codomain are the same, so we must show injectivity:

Claim: $\mathcal{L}$ is injective

Proof. Let $H = \sum_i L_i \otimes N_i$ be an element in $\mathrm{Hom}(U) \otimes \mathrm{Hom}(V)$; we need to check if $\mathcal{L}(H) = 0 \implies H = 0$

Lemma 1. For a linear map f, injectivity is equivalent to $f(x) = 0 \implies x = 0$

Proof. Injectivity is defined as $f(b) = f(a) \implies b = a$. This is equivalent to $f(b) - f(a) = 0 \implies b - a = 0$. Since f is linear, this is equivalent to $f(b-a) = 0 \implies b - a = 0$; thus, injectivity is equivalent to $\forall b, a : f(b-a) = 0 \implies b - a = 0$. □

So we will start with the assumption that $\mathcal{L}(H) = \sum_i (L_i \otimes N_i)(u \otimes v) = 0$ and attempt to work our way down to the injectivity conclusion.

From here, know that the tensor product of linear maps is itself a linear map; in other words, we can rewrite $\sum_i (L_i \otimes N_i)(u \otimes v) = \sum_i L_i(u) \otimes N_i(v)$. This is equivalent to element H applied to the arbitrary vector (u, v). In this representation, note that whether or not this linear combination is equal to zero depends directly on whether or not the sets $L_i(u); i = 1, ..., n$ and $N_j(v); j = 1, ..., n$ are linearly dependent:

Lemma 2. If both $L_i(u); i = 1, ..., n$ and $N_j(v); j = 1, ..., n$ are linearly independent then the sum $\sum_i L_i(u) \otimes N_i(v) \neq 0$ for any (u, v).

Proof. If both of those sets are linearly independent then the tensor product set of all $L_i(u) \otimes N_i(v)$ must also be independent. Thus the sum of elements in the tensor product set $\sum_i L_i(u) \otimes N_i(v) \neq 0$ by definition of independence, for any (u, v). □

The payoff is that if either one of L_i or N_i is linearly dependent then the map $\mathcal{L}$ is injective.

We will attempt to show that indeed one of the sets L_i and N_i have to be linearly dependent. First, if L_i were all linearly independent for i from 1 to n, by examining the sum of tensor products the only way the sum could be zero is for the linear combination to be the trivial linear combination- this implies that N_i must be dependent. Continuing case-by-case, let's assume without loss of generality that $L_1, L_2...L_m$ is maximally independent (that is, that L_i is not fully independent) for some m from 1 to $n-1$. Now:

$$\begin{aligned} 0 &= \sum_1^n L_i(u) \otimes N_i(v) \\ &= \sum_1^m L_i(u) \otimes N_i(v) + \sum_{m+1}^n L_i(u) \otimes N_i(v) \end{aligned}$$

Think about what the element L_{m+1} is: because of the maximal independent condition we are able to rewrite this element as the linear combination:

$$\begin{aligned} L_{m+1}(u) &= \sum_1^m \alpha_i L_i(u) \\ 0 &= \sum_1^m L_i(u) \otimes N_i(v) + L_{m+1}(u) \otimes N_{m+1}(u) + \sum_{m+2}^n L_i(u) \otimes N_i(v) \\ &= \sum_1^m L_i(u) \otimes N_i(v) + \sum_1^m \alpha_i L_i(u) \otimes N_{m+1}(v) + \sum_{m+2}^n L_i(u) \otimes N_i(v) \end{aligned}$$

We can now group the first two summands together:

$$\sum_1^m L_i(u) \otimes (N_i(v) + \alpha_i N_{m+1}(v)) + \sum_{m+2}^n L_i(u) \otimes N_i(v)$$

So our linear combination from above is

$$= \sum_1^m L_i(u) \otimes (N_i + \alpha_i N_{m+1})(v) + \sum_{m+2}^n L_i(u) \otimes N_i(v)$$

What we have accomplished is we have taken the element correspondin to L_{m+1} and rewritten it so that it is no longer present in the sum. In return, we have expanded the N_i terms. We can repeat this argument, compressing all the further terms above L_m. Finally, we arrive at $\sum_1^m L_i(u) \otimes \tilde{N}_i(v)$. Since the L_i were independent from 1 to m, in order for this sum to equal zero we must have that the $\tilde{N}_i$'s must be a dependent set (i from 1 to m).

However: with this in hand we can just repeat the argument, on $\tilde{N}_i$! Since they are a dependent set, without loss of generality we can say $\tilde{N_m}$ is capable of being written as a linear combination of elements $\tilde{N}_i; i = 1, ..., m-1$; $\tilde{N}_m = \sum_1^{m-1} \beta_j \tilde{N}_j(v)$; follow the theme of the previous argument:

$$\begin{aligned} 0 = \sum_1^m L_i(u) \otimes \tilde{N}_i(v) &= \sum_1^{m-1} L_i(u) \otimes \tilde{N}_i(v) + L_m(u) \otimes \left(\sum_1^{m-1} \beta_j \tilde{N}_j(v) \right) \\ &= \sum_1^{m-1} L_i(u) \otimes \tilde{N}_i(v) + \sum_1^{m-1} \beta_j L_m(u) \otimes \tilde{N}_j(v) \\ &= \sum_1^{m-1} (L_i + \beta_i L_m)(u) \otimes \tilde{N}_i(v) \end{aligned}$$

We have now rewritten the entire sum $0 = \sum L_i(u) \otimes N_i(v)$ in terms of $m-1$ elements, where each element is itself a linear combination of terms inside the set L_i for $i = 1, ..., m$; at this stage, we can keep repeating via induction to arrive at a single term of N and a long linear combination of elements from L_i for $i = 1, ..., m$- however, since we assumed that L_i is maximally independent for m, this is a contradiction since the whole thing sums up to 0. Thus we can conclude that the set L_i must be linearly dependent (since it cannot be maximally independent in any size, except 0). And thus the map $\mathcal{L}$ is injective. □

2 Quantum Mechanics

I'm not really sure about this section- I have never studied this part of QM theory before

The possible states of a quantum mechanical system occupy points in a Hilbert space, the "state space". The nature of the system itself depends on the problem in question; for a concrete example, we can describe the space given by states of various locations and momentums of particles; (x, ρ) where x is the location, ρ is the momentum. The "observables" are self adjoint bounded projections on the Hilbert space; we can tensor the spaces of several systems together to get the composite system made up of all the individual systems. In the concrete example we can tensor the Laplacian potential function $H = H_1 \otimes 1 + 1 \otimes H_2$; if we have n particles then the composite system will be $\otimes_1^n H_i$.

One of the important operators is the annihilation/creation operator pair. With n particles, the creation operator that governs the creation of a new particle is given by

$$\alpha^* : H_n = \otimes_1^n H_i \to H_{n+1}$$

$$\alpha^*(h) \left(v_1 \otimes \cdots \otimes v_n\right)$$
$$\alpha^*(h) = \sqrt{n+1} \left(h \otimes v_1 \otimes \cdots \otimes v_n\right)$$

h is the term corresponding to energy being absorbed and turned into a particle of the system, and $\sqrt{n+1}$ is the fudge factor.
The annihilator is the opposite: for an n particle system we have the map

$$\alpha : H_n \to H_{n-1}$$

$$\alpha(h) \left(v_1 \otimes \cdots \otimes v_n\right) = \sqrt{n} \cdot \sigma(h, v_1) \left(v_2 \otimes \cdots \otimes v_n\right)$$

Here, σ is the Hilbert space inner product; $\sqrt{n}$ is the fudge factor again, and h is the energy.

$$\|\alpha^*(h)\| \leqslant \sqrt{n+1}\|h\|$$

This implies this is a bounded linear operator.

The different energy states of a given quantum system are in fact the eigenvalues of the Hilbert space transforms.

3 A Concrete Example

Consider the space $m_2(\mathbb{R})$ of 2 by 2 matricies with real valued entries. The basis for this space is $u_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $u_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $u_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $u_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.
Consider the basis of $m_2(\mathbb{R})^*$, or all possible linear transformations on 2 by 2 matrices: $L_{ij}(u_i) = u_j$. These are all the different ways any entry in the matrix can be sent to any other matrix; an arbitrary linear transformation on 2 by 2 matrices will be the sum $\sum_{i,j=1}^{4} \alpha_{ij} L_{ij}$.
For example, we could have the linear transformation $L = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$ which would line up with the sum $2L_{13} + 2L_{24} + L_{31} + L_{42}$. Now, through the tensor product we can actually identify: $L_{ij} \approx u_i \otimes u_j$; thus the linear combination $\sum \alpha_{ij} L_{ij} = \sum \alpha_{ij}(u_i \otimes u_j)$, leading us to conclude that the *matrix representation* of a tensor product is the matrix of values $[\alpha_{ij}]$.
In the most explicit terms possible, here is a decomposition of a linear transformation into terms of tensor products and coefficients (using the linear transformation $L = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$ from before):

$$\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \cong 2L_{13} + 2L_{24} + L_{31} + L_{41} = \begin{pmatrix} 0L_{11} & 0L_{21} & 1L_{31} & 0L_{41} \\ 0L_{12} & 0L_{22} & 0L_{32} & 1L_{42} \\ 2L_{13} & 0L_{23} & 0L_{33} & 0L_{43} \\ 0L_{14} & 2L_{24} & 0L_{34} & 0L_{44} \end{pmatrix}$$

$$= \begin{pmatrix} 0L_{11} & 0L_{21} & 1L_{31} & 0L_{41} \\ 0L_{12} & 0L_{22} & 0L_{32} & 0L_{42} \\ 0L_{13} & 0L_{23} & 0L_{33} & 0L_{43} \\ 0L_{14} & 0L_{24} & 0L_{34} & 0L_{44} \end{pmatrix} + \begin{pmatrix} 0L_{11} & 0L_{21} & 0L_{31} & 0L_{41} \\ 0L_{12} & 0L_{22} & 0L_{32} & 1L_{42} \\ 0L_{13} & 0L_{23} & 0L_{33} & 0L_{43} \\ 0L_{14} & 0L_{24} & 0L_{34} & 0L_{44} \end{pmatrix} + 2\begin{pmatrix} 0L_{11} & 0L_{21} & 0L_{31} & 0L_{41} \\ 0L_{12} & 0L_{22} & 0L_{32} & 0L_{42} \\ 1L_{13} & 0L_{23} & 0L_{33} & 0L_{43} \\ 0L_{14} & 0L_{24} & 0L_{34} & 0L_{44} \end{pmatrix} +$$
$$2\begin{pmatrix} 0L_{11} & 0L_{21} & 0L_{31} & 0L_{41} \\ 0L_{12} & 0L_{22} & 0L_{32} & 0L_{42} \\ 0L_{13} & 0L_{23} & 0L_{33} & 0L_{43} \\ 0L_{14} & 1L_{24} & 0L_{34} & 0L_{44} \end{pmatrix}$$

Or for clarity: $= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} +$
$2\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = A + B + 2C + 2D$

Now: $u_2 \otimes u_1 = \begin{pmatrix} 0u_1 & 1u_1 \\ 0u_1 & 0u_1 \end{pmatrix} = A$; $u_2 \otimes u_4 = B$; $u_3 \otimes u_1 = C$; $u_3 \otimes u_4 = D$. The discrepancy in indicies (for example $(u_2 \otimes u_1) \cong L_{31}$) is easily handled by relabelling; $(u_i \otimes u_j)$ is, in fact, isomorphic to $\{L_{ij}\}$